SIAM J. CONTROL OPTIM.
Vol. 60, No. 5, pp. 2947–2970

CARLEMAN ESTIMATES OF REFINED STOCHASTIC BEAM EQUATIONS AND APPLICATIONS*

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Abstract. This paper is devoted to establishing global Carleman estimates for refined stochastic beam equations. First, by establishing a fundamental weighted identity, two Carleman estimates are derived with different weight functions for the refined stochastic beam equation, which is a coupled system consisting of a stochastic ordinary differential equation and a stochastic partial differential equation. As applications of these Carleman estimates, the exact controllability of the refined system is proved by the least controls in some sense. Different from classical stochastic beam equations, the refined one is exactly controllable at any time. Meanwhile, the uniqueness of an inverse source problem for refined stochastic beam equations is obtained without any requirement on the initial and terminal data.

Key words. refined stochastic beam equation, Carleman estimate, exact controllability, inverse problem

MSC codes. 93B05, 93B07

DOI. 10.1137/21M1463513

1. Introduction.

Let $T > 0$, let $Q = (0,1) \times (0,T)$, and let G_0 be a nonempty open subset of $(0,1)$. Fix a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ on which a one-dimensional standard Brownian motion $\{W(t)\}_{t\geq 0}$ is defined such that $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration generated by $W(\cdot)$, augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}$. Let $\mathcal{H}$ be a Banach space, and let $L^2_{\mathbb{F}}(0,T;\mathcal{H})$ be the Banach space consisting of all $\mathcal{H}$-valued $\mathbb{F}$-adapted processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{L^2(0,T;\mathcal{H})}) < \infty$; $L^\infty_{\mathbb{F}}(0,T;\mathcal{H})$ denotes the Banach space consisting of all $\mathcal{H}$-valued and $\mathbb{F}$-adapted essentially bounded processes; and $C_{\mathbb{F}}([0,T];L^r(\Omega;\mathcal{H}))$ denotes the Banach space consisting of all $\mathbb{F}$-adapted processes $X(\cdot)$ such that $X(\cdot) : [0,T] \to L^r_{\mathcal{F}_T}(\Omega;\mathcal{H})$ is continuous ($r \in [1,\infty]$). Similarly, one can define $C^m_{\mathbb{F}}([0,T];L^r(\Omega;\mathcal{H}))$ for any positive integer m. All of the above spaces are endowed with their canonical norms.

Beam is a kind of special structure widely existing in material mechanics and engineering mechanics, such as railway track, continuously supported piles, bridge support structure, and slender wings of aircraft. Study of beam models may date back to the 18th century, when Bernoulli found that the curvature of an elastic beam at any point is proportional to the bending moment and Euler (see [26]) studied the deformation of elastic beams under different load conditions. Since the Euler–Bernoulli beam theory has many applications in practical engineering problems, research in this field has received lots of attention [23, 25, 27].

*Received by the editors December 6, 2021; accepted for publication (in revised form) July 27, 2022; published electronically October 4, 2022.
https://doi.org/10.1137/21M1463513
Funding: This work was supported by the National Key R&D Program of China under grant 2018YFA0703800, by the National Natural Science Foundation of China under grant 61877057, and by the Postdoctoral Science Foundation of China under grant 2021TQ0353.
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In order to simulate the large amplitude vibration of an elastic panel excited by aerodynamic forces, the authors of [3] introduced a stochastic Euler–Bernoulli beam equation in which a force caused by random fluctuations was considered. The classical stochastic beam equation only involves random perturbations of external forces, which has the following form in one dimension:

$$
\begin{cases}
\mathrm{d}y_t + y_{xxxx}\mathrm{d}t = f\mathrm{d}t + g\mathrm{d}W(t) & \text{in } Q,\\
y(0,t)=0,\ y_x(0,t)=0 & \text{on } (0,T),\\
y(1,t)=0,\ y_x(1,t)=0 & \text{on } (0,T),\\
y(x,0)=y_0(x), \quad y_t(x,0)=y_1(x) & \text{in } (0,1),
\end{cases}
\tag{1.1}
$$

where y denotes the lateral displacement of the beam, (y_0, y_1) is the initial data, f denotes the continuous excitation, and g denotes the random perturbation.

In reality problems, due to air turbulence at high speed, both the pressure and the aerodynamic force are perturbed by random fluctuations. However, the random perturbation between velocity and displacement is ignored in the derivation of the classical stochastic beam equation (1.1). According to the dynamical theory of Brownian motions in [21] and the derivation process of the model in [17], we make a modification to the classical stochastic beam equation. Stimulated by the uncertainty between velocity and displacement, and combining with the classical Euler–Bernoulli beam theory, we can get the following refined stochastic beam equation:

$$
\begin{cases}
\mathrm{d}y = \hat{y}\mathrm{d}t + \tilde{f}\mathrm{d}W(t) & \text{in } Q,\\
\mathrm{d}\hat{y} + y_{xxxx}\mathrm{d}t = f\mathrm{d}t + g\mathrm{d}W(t) & \text{in } Q,\\
y(0,t)=0,\ y_x(0,t)=0 & \text{on } (0,T),\\
y(1,t)=0,\ y_x(1,t)=0 & \text{on } (0,T),\\
y(x,0)=y_0(x), \quad \hat{y}(x,0)=\hat{y}_0(x) & \text{in } (0,1),
\end{cases}
$$

where y denotes the lateral displacement of the beam, $\hat{y}$ denotes the lateral velocity of the beam, $(y_0, \hat{y}_0)$ is the initial data, $\tilde{f}$ denotes the uncertainty between velocity and displacement, f denotes the continuous excitation, and g denotes the random perturbations.

The Carleman-type estimate was first introduced by Carleman in 1939 to study the uniqueness for elliptic equations in two dimensions [2]. It has become an important tool in studying the uniqueness, control, and inverse problems for deterministic partial differential equations (see [1, 6, 10, 11, 28, 32] and the references therein). However, people know little about its stochastic counterpart. We refer the reader to [5, 14, 24, 30, 31] for some known Carleman estimates of the stochastic partial differential equations.

This paper is devoted to establishing global Carleman estimates for refined stochastic beam equations and applying these estimates to study two classes of important ill-posed problems: the exact controllability and inverse source problems for refined stochastic beam equations.

It is well known that a deterministic beam equation is exactly controllable by applying controls at the boundary or inside. But, we show that the exact controllability of the classical stochastic beam equation is to fail at any time, even if controls are acted everywhere on the drift term, the diffusion term, and the boundary. Note that these controls are the most powerful control actions that one can introduce into an equation. Once the model is reasonably modified, we can prove the exact controllability by the Carleman estimate.

As another application of Carleman estimates, we also study an inverse source problem for refined stochastic beam equations. In [30], Carleman estimates of classical stochastic beam equations are established to study the inverse source problem. Compared with [30], we choose weight functions with singularity in the Carleman estimate. By this estimate, we can determine multiple source terms simultaneously with less observation information.

There are numerous works (see [9, 12, 13, 19, 20, 29] and the references therein) addressing controllability and inverse source problems for deterministic beam equations. However, new difficulties arise in dealing with the related control and inverse problem of the stochastic counterpart; for example, the solution of a stochastic partial differential equation is non-differentiable with respect to the noise variable, and the usual compactness embedding result is not valid.

The contribution and findings of this work are summarized as follows:

• We establish Carleman estimates for the refined stochastic beam equations. By these estimates, we reveal the difference in controllability between the stochastic beam equation and the refined equation.

• We prove the exact controllability for the refined equation with a minimum number of controls and show that the classical stochastic beam equation is not exactly controllable at any time.

• We establish the uniqueness of an inverse source problem for refined stochastic beam equations without any requirement on initial and terminal data. The inverse problem has determined multiple source terms simultaneously, not only the drift source terms but also the diffusion terms.

The rest of this paper is organized as follows. Section 2 presents the main results in this paper. In section 3, we give a pointwise weighted identity for the refined stochastic beam operator. Section 4 is devoted to proving the exact controllability results for the refined stochastic beam equation, while section 5 gives a detailed proof of the inverse problem result. Finally, we summarize the paper and discuss future topics that are worth investigating in section 6.

For notational simplicity, we give some notations:

$$
\begin{aligned}
&\mathcal{H}_1 = (H^4(0,1)\cap H_0^2(0,1))\times H_0^2(0,1),\ \mathcal{H}_2 = L^2(0,1)\times H^{-2}(0,1),\\
&\mathcal{H}_3 = C_{\mathbb{F}}([0,T];L^2(\Omega;L^2(0,1)))\cap C_{\mathbb{F}}^1([0,T];L^2(\Omega;H^{-2}(0,1))),\\
&\mathcal{H}_4 = C_{\mathbb{F}}([0,T];L^2(\Omega;L^2(0,1)))\times C_{\mathbb{F}}([0,T];L^2(\Omega;H^{-2}(0,1))),\\
&\mathcal{H}_5 = L^2_{\mathbb{F}}(\Omega;C([0,T];H^4(0,1)\cap H_0^2(0,1)))\times L^2_{\mathbb{F}}(\Omega;C([0,T];H_0^2(0,1))),\\
&\mathcal{H}_6 = C_{\mathbb{F}}([0,T];L^2(\Omega;L^2(0,1)))\times L^2_{\mathbb{F}}(0,T;L^2(0,1))\\
&\qquad \times C_{\mathbb{F}}([0,T];L^2(\Omega;H^{-2}(0,1)))\times L^2_{\mathbb{F}}(0,T;H^{-2}(0,1)).
\end{aligned}
$$

2. Main results. Consider the following refined stochastic beam equation:

$$
\begin{cases}
\mathrm{d}y = \big(\hat{y}+f_1\big)\mathrm{d}t + g_1\mathrm{d}W(t) & \text{in } Q,\\
\mathrm{d}\hat{y} + y_{xxxx}\mathrm{d}t = f_2\mathrm{d}t + g_2\mathrm{d}W(t) & \text{in } Q,\\
y(0,t) = y_x(0,t) = 0 & \text{on } (0,T),\\
y(1,t) = y_x(1,t) = 0 & \text{on } (0,T),\\
y(x,0) = y_0(x), \quad \hat{y}(x,0) = \hat{y}_0(x) & \text{in } (0,1),
\end{cases}
\tag{2.1}
$$

where $(y_0,\hat{y}_0)\in\mathcal{H}_1$, $f_1,f_2,g_2\in L^2_{\mathbb{F}}(0,T;H_0^2(0,1))$, and $g_1\in L^2_{\mathbb{F}}(0,T;H^4(0,1)\cap H_0^2(0,1))$.

Similar to the discussion in [4, 18], for any initial data $(y_0, \hat{y}_0) \in \mathcal{H}_1$, $f_1, f_2, g_2 \in L^2_{\mathbb{F}}(0,T; H^2_0(0,1))$, and $g_1 \in L^2_{\mathbb{F}}(0,T; H^4(0,1) \cap H^2_0(0,1))$, (2.1) admits a unique solution $(y, \hat{y}) \in \mathcal{H}_5$.

Here, we introduce some auxiliary functions. Let G_0 and G_1 be nonempty open subsets of $(0,1)$ satisfying $\overline{G_1} \subseteq G_0$. Let $\psi_1, \psi_2 \in C^4([0,1])$ satisfy that

$$\psi_1(x) > 0 \text{ in } [0,1], \quad \psi_{1,x}(x) < 0 \text{ in } [0,1]$$

and

$$\psi_2(x) > 0 \text{ in } (0,1), \quad \psi_2(0) = \psi_2(1) = 0, \quad |\psi_{2,x}| > 0 \text{ in } (0,1) \setminus G_1.$$

For any parameters $\lambda, \mu \geq 1$ and $i = 1, 2$, put

$$\theta_i = e^{\ell_i}, \quad \ell_i = \lambda \eta_i, \quad \eta_i = \frac{e^{\mu\psi_i} - e^{2\mu|\psi_i|_{C([0,1])}}}{t^2(T-t)^2}, \quad \varphi_i = \frac{e^{\mu\psi_i}}{t^2(T-t)^2}.$$

By establishing a suitable weighted identity and choosing a weight function $\theta_1 = \theta_1(x,t)$, then we have the following global Carleman estimate for (2.1).

THEOREM 2.1. *There are two positive constants μ_1 and $\lambda_1(\mu)$, such that for all $\mu \geq \mu_1$, $\lambda \geq \lambda_1(\mu)$, and any solution $(y, \hat{y}) \in \mathcal{H}_5$ to* (2.1)*, it holds that*

(2.2)
$$\begin{aligned}
&\mathbb{E}\int_Q \theta_1^2\Big[\lambda^7\mu^8\varphi_1^7 y^2 + \lambda^5\mu^6\varphi_1^5 y_x^2 + \lambda^3\mu^4\varphi_1^3\big(y_{xx}^2 + \hat{y}^2\big) + \lambda\mu^2\varphi_1\big(y_{xxx}^2 + \hat{y}_x^2\big)\Big]dx\,dt \\
&\leq C\mathbb{E}\int_Q \theta_1^2\big(\lambda^6\mu^6\varphi_1^6 f_1^2 + \lambda^4\mu^4\varphi_1^4 f_{1,x}^2 + \lambda^2\mu^2\varphi_1^2 f_{1,xx}^2 + f_2^2 \\
&\qquad + \lambda^6\varphi_1^6 g_1^2 + \lambda^4\varphi_1^4 g_{1,x}^2 + \lambda^2\varphi_1^2 g_{1,xx}^2 + g_{1,xxx}^2 + \lambda^2\mu^8\varphi_1^2 g_2^2\big)\,dx\,dt \\
&\qquad + C\mathbb{E}\int_0^T \theta_1^2\Big[\lambda\mu\varphi_1 y_{xxx}^2(0,t) + \lambda^3\mu^3\varphi_1^3 y_{xx}^2(0,t)\Big]dt.
\end{aligned}$$

Proof. See Appendix B. ∎

Moreover, the other global Carleman estimate for (2.1) is established by taking different weight function $\theta_2 = \theta_2(x,t)$.

THEOREM 2.2. *There are two positive constants μ_2 and $\lambda_2(\mu)$, such that for all $\mu \geq \mu_2$, $\lambda \geq \lambda_2(\mu)$, and any solution $(y, \hat{y}) \in \mathcal{H}_5$ to* (2.1)*, it holds that*

(2.3)
$$\begin{aligned}
&\mathbb{E}\int_Q \theta_2^2\Big[\lambda^7\mu^8\varphi_2^7 y^2 + \lambda^5\mu^6\varphi_2^5 y_x^2 + \lambda^3\mu^4\varphi_2^3\big(y_{xx}^2 + \hat{y}^2\big) + \lambda\mu^2\varphi_2\big(y_{xxx}^2 + \hat{y}_x^2\big)\Big]dxdt \\
&\leq C\mathbb{E}\int_Q \theta_2^2\big(\lambda^6\mu^6\varphi_2^6 f_1^2 + \lambda^4\mu^4\varphi_2^4 f_{1,x}^2 + \lambda^2\mu^2\varphi_2^2 f_{1,xx}^2 + f_2^2 \\
&\qquad + \lambda^6\varphi_2^6 g_1^2 + \lambda^4\varphi_2^4 g_{1,x}^2 + \lambda^2\varphi_2^2 g_{1,xx}^2 + g_{1,xxx}^2 + \lambda^2\mu^8\varphi_2^2 g_2^2\big)\,dx\,dt \\
&\qquad + C\mathbb{E}\int_0^T\int_{G_0} \theta_2^2\big(\lambda^7\mu^8\varphi_2^7 y^2 + \lambda^5\mu^6\varphi_2^5 y_x^2 + \lambda^3\mu^4\varphi_2^3 y_{xx}^2 + \lambda\mu^2\varphi_2 y_{xxx}^2\big)dxdt.
\end{aligned}$$

Proof. See Appendix C. ∎

Remark 2.3. By choosing different weighted functions, two Carleman estimates (2.2) and (2.3) are established. The main difference between the above estimates is the local information on the right-hand sides of the inequalities: one is internal information, and the other is boundary information.

Remark 2.4. Compared with the results in [30], there are no terms with respect to initial and terminal data on the right-hand sides of (2.2) and (2.3). This means that the additional conditions on the initial and terminal data are no longer required by the above Carleman estimates, and therefore the known observations may be reduced. The main reason is the choice of the "singular" weight function θ_i with respect to time.

The first application of the above Carleman estimates is the exact controllability of the refined stochastic beam equation. Controllability means to find a control such that the corresponding state of the considered system achieves a prescribed goal at a given time. Consider the following controlled stochastic beam equation:

$$(2.4)\qquad \begin{cases} \mathrm{d}y = \hat{y}\mathrm{d}t + f\mathrm{d}W(t) & \text{in } Q, \\ \mathrm{d}\hat{y} + y_{xxxx}\mathrm{d}t = \big(a_1 y + a_2 y_x + a_3 g\big)\mathrm{d}t + g\mathrm{d}W(t) & \text{in } Q, \\ y(0,t) = h_1,\ y_x(0,t) = h_2 & \text{on } (0,T), \\ y(1,t) = h_3,\ y_x(1,t) = h_4 & \text{on } (0,T), \\ y(x,0) = y_0(x), \quad \hat{y}(x,0) = \hat{y}_0(x) & \text{in } (0,1), \end{cases}$$

where $(y_0, \hat{y}_0)$ is the initial data and the coefficients $a_1, a_3 \in L^\infty(0,T; L^\infty(0,1))$, $a_2 \in L^\infty(0,T; W^{1,\infty}(0,1))$, $f \in L^2_{\mathbb{F}}(0,T; L^2(0,1))$, $g \in L^2_{\mathbb{F}}(0,T; H^{-2}(0,1))$, and $h_i \in L^2_{\mathbb{F}}(0,T)$ $(i = 1,2,3,4)$ are controls.

Similar to the method used in [17], for any $(y_0, \hat{y}_0) \in \mathcal{H}_2$, $f \in L^2_{\mathbb{F}}(0,T; L^2(0,1))$, and $g \in L^2_{\mathbb{F}}(0,T; H^{-2}(0,1))$, (2.4) admits a unique transposition solution $(y, \hat{y}) \in \mathcal{H}_4$ (the definition of transposition solution can be found in [17]).

Remark 2.5. The term $a_3 g$ in (2.4) reflects the effect of the control g in the diffusion term to the drift term in some way. It is the side effect of the control g. In other words, if a control g is put in the diffusion term, then $a_3 g$ appears passively in the drift term.

Now we give the definition of the exact controllability for (2.4).

DEFINITION 2.6. *The system* (2.4) *is called exactly controllable at time T if for any $(y_0, \hat{y}_0) \in \mathcal{H}_2$ and $(y_1, \hat{y}_1) \in L^2_{\mathcal{F}_T}(\Omega; L^2(0,1)) \times L^2_{\mathcal{F}_T}(\Omega; H^{-2}(0,1))$ one can find a pair of controls*

$$(f, g, h_1, h_2) \in L^2_{\mathbb{F}}(0,T; L^2(0,1)) \times L^2_{\mathbb{F}}(0,T; H^{-2}(0,1)) \times (L^2_{\mathbb{F}}(0,T))^2,$$

such that the solution $(y, \hat{y})$ of (2.4) *corresponding to the above controls and $h_3 = h_4 = 0$ satisfies that $(y(\cdot,T), \hat{y}(\cdot,T)) = (y_1, \hat{y}_1)$.*

There are some controllability results for stochastic systems [5, 7, 8, 15, 17, 24]. Similar to the results in [17], we find that the exact controllability of the classical stochastic beam system fails for any $T > 0$, even if the controls are acted everywhere on the whole domain $(0,1)$ and the boundary. This is quite different from the well-known controllability property of the deterministic beam system. But, after reasonable modifications of the stochastic beam system, the following exact controllability result is obtained.

THEOREM 2.7. *The system* (2.4) *is exactly controllable at any time $T > 0$.*

By the classical duality argument [17], we transform the above null controllability into an appropriate observability estimate (see Proposition 4.6 given below) and give the proof of the theorem in section 4.

Remark 2.8. The boundary controls h_1 and h_2 in (2.4) are imposed on the boundary point 0. The above result still holds if they are effective on the boundary point 1 for (2.4).

Remark 2.9. By means of the Carleman estimate (2.3), the exact controllability of (2.4) can also be established if the controls f, g are acted in the diffusion terms and the local internal control $h_5 \in L^2_{\mathbb{F}}(0,T;H^{-3}(G_0))$ is put in the drift term for the second equation of (2.4).

Remark 2.10. It is worth noting that four controls f and g (in the diffusion terms) and h_1 and h_2 (on the boundary) are required in (2.4) to derive the exact controllability. Moreover, the controls f and g in diffusion terms are active on the whole domain. In fact, these conditions cannot be weakened. More details will be given in Theorem 4.8.

Remark 2.11. Compared with [17], the background of our problem is different. In [17], the exact controllability of the stochastic wave equation is investigated. In this paper, different weight functions with singularity are chosen, and the Carleman estimates containing local interior and boundary information are established. By the Carleman estimates, not only is the exact controllability of the refined stochastic beam equations studied but also an inverse source problem.

The other application of the Carleman estimates in this paper is the inverse source problem for the following refined stochastic beam equation:

$$
(2.5)\qquad \begin{cases} \mathrm{d}y = \Big[\hat{y} + A(t)R(x,t)\Big]\mathrm{d}t + \Big[c_1 y + B(t)R(x,t)\Big]\mathrm{d}W(t) & \text{in } Q,\\ \mathrm{d}\hat{y} + y_{xxxx}\mathrm{d}t = \Big[c_2 y + c_3 y_x + H(t)R(x,t)\Big]\mathrm{d}t & \\ \qquad\qquad + \Big[c_4 y + P(t)R(x,t)\Big]\mathrm{d}W(t) & \text{in } Q,\\ y(0,t) = y_x(0,t) = 0 & \text{on } (0,T),\\ y(1,t) = y_x(1,t) = 0 & \text{on } (0,T),\\ y(x,0) = y_0(x), \quad \hat{y}(x,0) = \hat{y}_0(x) & \text{in } (0,1), \end{cases}
$$

where the initial data $(y_0, \hat{y}_0) \in \mathcal{H}_1$, and the coefficients $c_1 \in L^\infty_{\mathbb{F}}(0,T;W^{4,\infty}(0,1))$, $c_2, c_4 \in L^\infty_{\mathbb{F}}(0,T;W^{2,\infty}(0,1))$, $c_3 \in L^\infty_{\mathbb{F}}(0,T;W^{3,\infty}(0,1))$, and $A, B, H, P \in L^2_{\mathbb{F}}(0,T)$ are four unknown sources.

For suitably given function R, our inverse problem is to determine four source functions $A(\cdot)$, $B(\cdot)$, $H(\cdot)$, and $P(\cdot)$ by means of the known observation information of the boundary and the interior domain. The main result on the uniqueness of (2.5) is as follows.

THEOREM 2.12. *Assume that* $R \in C^5(\overline{Q})$ *and* $R \neq 0$ *in* $\overline{Q}$. *Let* $(y, \hat{y}) \in \mathcal{H}_5$ *be any solution to* (2.5). *If*

$$
(2.6)\qquad \begin{cases} y_{xx}(0,t) = y_{xx}(1,t) = 0 & \text{on } (0,T),\\ y = 0 & \text{in } G_0 \times (0,T), \end{cases}
$$

then

$$
A(t) = B(t) = H(t) = P(t) = 0 \quad \text{for all} \quad t \in [0,T], \ \mathbb{P}\text{-}a.s.
$$

The key to obtaining the above result is the Carleman estimate, and the proof of the theorem is given in section 5.

Remark 2.13. Compared with [30], by choosing the singular weight functions, we establish the different Carleman estimates. From the Carleman estimate, we can determine four sources simultaneously, not only the drift source terms but also the diffusion terms. And there is no restriction $\sqrt{T-t}$ in the right-hand side of the diffusion source term. Furthermore, the requirements for initial and final data are removed from the observation information.

Remark 2.14. In (2.5), all sources have the form of separated variables. If the sources in the drift term are general, it is easy to show there are counterexamples (see [16]). It is interesting to consider the general sources in the diffusion term. More precisely, in (2.5), $B(t)R(x,t)$ and $P(t)R(x,t)$ are replaced by $B(x,t)$ and $P(x,t)$, respectively. However, this remains to be done and it is our future work.

Remark 2.15. It seems a bit strange that the measurements in local domain G_0 and on the boundary are simultaneously needed in terms of mathematical theory, but it is acceptable in practical application. In the real wave models, internal information is not easy to obtain. However, it does not seem difficult for the real beam models. For example, we can put a tiny sensor in the wing of an airplane, or in a gap of the bridge structure, to measure the information inside a beam model.

Remark 2.16. In [30], the known observations are in the local area. If the known information is only on the local boundary, it is still open and we will consider it in a forthcoming paper.

3. A pointwise weighted identity. This section is devoted to establishing a weighted identity for stochastic beam-like operator $d\hat{y} + y_{xxxx}\,dt$, which will play a key role in what follows.

Assume that $\Psi \in C^3(\mathbb{R}^n \times (0,T))$, $\ell \in C^5(\mathbb{R}^n \times (0,T))$, and put

$$\begin{cases} \Psi = -9\ell_x^2\ell_{xx}, \\ \Phi = \ell_x^4 - 6\ell_x^2\ell_{xx} + 3\ell_{xx}^2 + 4\ell_x\ell_{xxx} - \ell_{xxxx} + \ell_t^2 - \ell_{tt}, \\ A = 3\big(\Psi\ell_x^2 - \Psi\ell_{xx}\big)_{xx} + \big(\ell_t\Phi - \ell_t\Psi\big)_t + 2\big(\ell_x\Phi - \ell_x\Psi\big)_{xxx} \\ \qquad + \Psi\big(\Phi - \Psi\big) + 2\Big[D\big(\Phi - \Psi\big)\Big]_x, \\ B = 12\Big[D\big(\ell_x^2 - \ell_{xx}\big)\Big]_x - 6\big(\Psi\ell_x^2 - \Psi\ell_{xx}\big) - 6\big(\ell_t\ell_x^2 - \ell_t\ell_{xx}\big)_t \\ \qquad - 6\big(\ell_x\Phi - \ell_x\Psi\big)_x, \\ D = \ell_x^3 - 3\ell_x\ell_{xx} + \ell_{xxx}. \end{cases}$$

LEMMA 3.1. *Let y be an $H^4(\mathbb{R})$-valued semimartingale, let $\hat{y}$ be an $H^2(\mathbb{R})$-valued semimartingale, and let*

$$dy = \hat{y}\,dt + f_1\,dt + g_1\,dW(t) \text{ in } Q \tag{3.1}$$

for some $f_1 \in L^2_{\mathbb{F}}(0,T;H^2_0(0,1))$, $g_1 \in L^2_{\mathbb{F}}(0,T;H^4(0,1) \cap H^2_0(0,1))$. Set $z = \theta y$,

$\hat{z} = \theta\hat{y} + \ell_t z$, *and* $\theta = e^{\ell}$. *Then, for a.e.* $x \in (0,1)$, *and* $\mathbb{P}$*-a.s.* $\omega \in \Omega$,

$$
\begin{aligned}
&\theta I\big(d\hat{y} + y_{xxxx}\,dt\big)\\
&= dM + V_x\,dt + I^2\,dt + 4\big(\ell_x \hat{z}\,dz_{xx}\big)_x + 6\ell_{xx}\hat{z}_x^2\,dt + 2\ell_{xx}z_{xxx}^2\,dt\\
&\quad + Az^2\,dt + Bz_x^2\,dt + U\,dt + P\,dt + \big(\ell_{tt} - 2\ell_{xxxx} - 2D_x - \Psi\big)\hat{z}^2\,dt\\
&\quad + \Big[\Psi - 6D_x + \ell_{tt} + 12\big(\ell_x^3 - \ell_x\ell_{xx}\big)_x\Big] z_{xx}^2\,dt + \ell_t\big(\Phi - \Psi\big)\big(dz\big)^2\\
&\quad - 6\ell_t\big(\ell_x^2 - \ell_{xx}\big)\big(dz_x\big)^2 + \ell_t\big(dz_{xx}\big)^2 + \ell_t\big(d\hat{z}\big)^2 - \Psi\,dz\,d\hat{z}\\
&\quad + 4\ell_x\,d\hat{z}\,dz_{xxx} + 4D\,d\hat{z}\,dz_x + \Big\{\Big[2\ell_t\big(\Phi - \Psi\big)z + \Psi\hat{z} - I\ell_t\Big]\theta g_1\\
&\quad + 4D\hat{z}\big(\theta g_1\big)_x - 12\big(\ell_t\ell_x^2 - \ell_t\ell_{xx}\big)z_x\big(\theta g_1\big)_x\\
&\quad + \Big[2\ell_t z_{xx} - 4\big(\ell_{xx}\hat{z} + \ell_x\hat{z}_x\big)\Big]\big(\theta g_1\big)_{xx}\Big\}\,dW(t),
\end{aligned}
\tag{3.2}
$$

where

$$
\left\{
\begin{aligned}
&I = -2\ell_t\hat{z} - 4\ell_x z_{xxx} - 4Dz_x + \Psi z,\\
&M = -\ell_t\hat{z}^2 - \big(4\ell_x z_{xxx} + 4Dz_x - \Psi z\big)\hat{z} - \ell_t z_{xx}^2 + 6\ell_t\big(\ell_x^2 - \ell_{xx}\big)z_x^2\\
&\qquad - \ell_t\big(\Phi - \Psi\big)z^2,\\
&V = -4\ell_{xx}\hat{z}\hat{z}_x - 2\ell_x\hat{z}_x^2 + 2\big(D + \ell_{xxx}\big)\hat{z}^2 - 2\ell_t\hat{z}z_{xxx} - 2\ell_x z_{xxx}^2\\
&\qquad - 4Dz_x z_{xxx} + \Psi z z_{xxx} + 2\Big[D - 6\big(\ell_x^3 - \ell_x\ell_{xx}\big)\Big]z_{xx}^2 + \big(4D_x - \Psi\big)z_x z_{xx}\\
&\qquad + 2\ell_t\hat{z}_x z_{xx} - 12\big(\ell_t\ell_x^2 - \ell_t\ell_{xx}\big)\hat{z}z_x - 4\big(\ell_x\Phi - \ell_x\Psi\big)zz_{xx}\\
&\qquad + 2\Big[\big(\ell_x\Phi - \ell_x\Psi\big) - 6D\big(\ell_x^2 - \ell_{xx}\big)\Big]z_x^2 + 8\big(\Psi\ell_x^2 - \Psi\ell_{xx}\big)zz_x\\
&\qquad + 4\big(\ell_x\Phi - \ell_x\Psi\big)_x zz_x - \Big[2D\big(\Phi - \Psi\big) + 3\big(\Psi\ell_x^2 - \Psi\ell_{xx}\big)_x\\
&\qquad + 2\big(\ell_x\Phi - \ell_x\Psi\big)_{xx}\Big]z^2,\\
&P = \big(\Psi_x - 4D_{xx}\big)z_x z_{xx} + \Big[4D_t + 12\big(\ell_t\ell_x^2 - \ell_t\ell_{xx}\big)_x\Big]\hat{z}z_x - \Psi_t\hat{z}z\\
&\qquad - 2\ell_{tx}\hat{z}_x z_{xx} + 6\ell_{xt}\hat{z}z_{xxx} - \Psi_x zz_{xxx},\\
&U = \Big[-I\ell_t + 2\big(\Phi - \Psi\big)\ell_t z - \Psi\hat{z}\Big]\theta f_1 + \Big[4D\hat{z} - 12\big(\ell_t\ell_x^2 - \ell_t\ell_{xx}\big)z_x\Big]\big(\theta f_1\big)_x\\
&\qquad + \Big[2\ell_t z_{xx} - 4\big(\ell_{xx}\hat{z} + \ell_x\hat{z}_x\big)\Big]\big(\theta f_1\big)_{xx},
\end{aligned}
\right.
$$

and $(dz)^2$, $(d\hat{z})^2$ *denote the quadratic variation processes of* z *and* $\hat{z}$, *respectively.*

Proof. See Appendix A. ∎

Then, from the weighted identity (3.2), and by properly choosing different weight functions, one can establish the global Carleman estimates (2.2) and (2.3). The detailed proofs of Theorems 2.1 and 2.2 are provided in Appendices B and C, respectively.

4. Controllability. In this section, we study the controllability problem for the classical and the refined stochastic beam systems. Fist, consider the following classical stochastic beam system:

$$
\begin{cases}
\mathrm{d}y_t + y_{xxxx}\mathrm{d}t = \big(a_1 y + a_2 y_x + f\big)\mathrm{d}t + g\mathrm{d}W(t) & \text{in } Q,\\
y(0,t) = h_1,\ y_x(0,t) = h_2 & \text{on } (0,T),\\
y(1,t) = h_3,\ y_x(1,t) = h_4 & \text{on } (0,T),\\
y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x) & \text{in } (0,1),
\end{cases}
\tag{4.1}
$$

where the initial data $(y_0, y_1) \in \mathcal{H}_2$, y is the state variable, and $a_1 \in L^\infty(0,T;L^\infty(0,1))$, $a_2 \in L^\infty(0,T;W^{1,\infty}(0,1))$, $f, g \in L^2_{\mathbb{F}}(0,T;H^{-2}(0,1))$, and $h_i \in L^2_{\mathbb{F}}(0,T)$ $(i = 1,2,3,4)$ are controls.

Similar to the method used in [17], for any initial data $(y_0, y_1) \in \mathcal{H}_2$, $f, g \in L^2_{\mathbb{F}}(0,T;H^{-2}(0,1))$, and $h_i \in L^2_{\mathbb{F}}(0,T)$ $(i = 1,2,3,4)$, (4.1) admits a unique solution $y \in \mathcal{H}_3$.

Recall the definition of the exact controllability for (4.1).

Definition 4.1. *The system* (4.1) *is called exactly controllable at time T if for any $(y_0, y_1) \in \mathcal{H}_2$ and $(y_0', y_1') \in L^2_{\mathcal{F}_T}(\Omega; L^2(0,1)) \times L^2_{\mathcal{F}_T}(\Omega; H^{-2}(0,1))$ one can find a pair of controls*

$$(f, g, h_1, h_2, h_3, h_4) \in L^2_{\mathbb{F}}(0,T;H^{-2}(0,1)) \times L^2_{\mathbb{F}}(0,T;H^{-2}(0,1)) \times (L^2_{\mathbb{F}}(0,T))^4,$$

such that the solution y to (4.1) *satisfies $(y(\cdot,T), y_t(\cdot,T)) = (y_0', y_1')$.*

Let us give the negative controllability result for (4.1).

Theorem 4.2. *The system* (4.1) *is not exactly controllable for any $T > 0$.*

The proof of Theorem 4.2 is similar to that of Theorem 2.1 in [17], which we omit here. We show that the classical stochastic system (4.1) is not exactly controllable at any time, even if the controls acted everywhere on the domain $(0,1)$ and boundary. Obviously, it is quite different from the well-known controllability results of deterministic beam systems.

Next, as an application of the Carleman estimate (2.2), the controllability problems of the refined system (2.4) is considered. According to the duality argument, the exact controllability problem of a stochastic partial differential equation is usually transformed into the observability estimate of the corresponding adjoint equation. But, generally speaking, it is easier to establish an observability estimate for a forward equation than to prove an observability estimate for a backward one in the stochastic counterparts. Then we transform the exact controllability problem of (2.4) into the exact controllability problem of a backward stochastic beam system. Therefore, the following controlled backward stochastic beam system is introduced:

$$(4.2)\qquad \begin{cases} \mathrm{d}p = \hat{p}\mathrm{d}t + P\mathrm{d}W(t) & \text{in } Q, \\ \mathrm{d}\hat{p} + p_{xxxx}\mathrm{d}t = \Big(a_1 p + a_2 p_x + a_3 \hat{P}\Big)\mathrm{d}t + \hat{P}\mathrm{d}W(t) & \text{in } Q, \\ p(0,t) = h_1(t), \quad p_x(0,t) = h_2(t) & \text{on } (0,T), \\ p(1,t) = 0, \quad p_x(1,t) = 0 & \text{on } (0,T), \\ p(x,T) = p_T(x), \quad \hat{p}(x,T) = \hat{p}_T(x) & \text{in } (0,1), \end{cases}$$

where $(p_T, \hat{p}_T) \in L^2_{\mathcal{F}_T}(\Omega; L^2(0,1)) \times L^2_{\mathcal{F}_T}(\Omega; H^{-2}(0,1))$, $(p, \hat{p}, P, \hat{P})$ are the states, (h_1, h_2) are controls, $a_1, a_3 \in L^\infty_{\mathbb{F}}(0,T;L^\infty(0,1))$, and $a_2 \in L^\infty_{\mathbb{F}}(0,T;W^{1,\infty}(0,1))$.

By [18], for any $(p_T, \hat{p}_T) \in L^2_{\mathcal{F}_T}(\Omega; L^2(0,1)) \times L^2_{\mathcal{F}_T}(\Omega; H^{-2}(0,1))$ and $(h_1, h_2) \in (L^2_{\mathbb{F}}(0,T))^2$, (4.2) admits a unique transposition solution $(p, \hat{p}, P, \hat{P}) \in \mathcal{H}_6$.

Now we give the definition of the exact controllability for (4.2).

Definition 4.3. *The system* (4.2) *is called exactly controllable at time T if for any $(p_T, \hat{p}_T) \in L^2_{\mathcal{F}_T}(\Omega; L^2(0,1)) \times L^2_{\mathcal{F}_T}(\Omega; H^{-2}(0,1))$ and $(p_0, \hat{p}_0) \in \mathcal{H}_2$ one can find a pair of controls*

$$(h_1, h_2) \in (L^2_{\mathbb{F}}(0,T))^2,$$

such that the solution $(p, \hat{p}, P, \hat{P})$ to (4.2) *satisfies $(p(\cdot,0), \hat{p}(\cdot,0)) = (p_0, \hat{p}_0)$.*

By Definitions 2.6 and 4.3, we have the following result concerning the relationship between solutions of (2.4) and (4.2).

PROPOSITION 4.4. *One the one hand, if* $(y, \hat{y})$ *is a solution of* (2.4)*, then*

$$(p, \hat{p}, P, \hat{P}) = (y, \hat{y}, f, g)$$

is a solution of (4.2) *with the final data* $(p_T, \hat{p}_T) = (y(T), \hat{y}(T))$*. On the other hand, if* $(p, \hat{p}, P, \hat{P})$ *is a solution of* (4.2)*, then*

$$(y, \hat{y}) = (p, \hat{p})$$

is a solution of (2.4) *with the initial data* $(y_0, \hat{y}_0) = (p(0), \hat{p}(0))$ *and nonhomogeneous terms* $(f, g) = (P, \hat{P})$*.*

From Proposition 4.4, the following result holds.

PROPOSITION 4.5. *The system* (2.4) *is exactly controllable at time* T *if and only if the system* (4.2) *is exactly controllable at time* T*.*

It is shown that the exact controllability problem of the forward stochastic beam system (2.4) can be transformed into the exact controllability problem of the backward system (4.2). By duality theory, in order to study the exact controllability of (4.2), introduce the following stochastic beam system:

$$\begin{cases} \mathrm{d}z = \hat{z}\mathrm{d}t - a_3 z\mathrm{d}W(t) & \text{in } Q, \\ \mathrm{d}\hat{z} + z_{xxxx}\mathrm{d}t = \Big[(a_1 - a_{2,x})z - a_2 z_x\Big]\mathrm{d}t & \text{in } Q, \\ z(0,t) = z_x(0,t) = 0 & \text{on } (0,T), \\ z(1,t) = z_x(1,t) = 0 & \text{on } (0,T), \\ z(x,0) = z_0(x), \quad \hat{z}(x,0) = \hat{z}_0(x) & \text{in } (0,1), \end{cases} \tag{4.3}$$

where $(z_0, \hat{z}_0) \in \mathcal{H}_1$.

The following result shows that the exact controllability of (4.2) is equivalent to an observability estimate of (4.3).

PROPOSITION 4.6. *The system* (4.2) *is exactly controllable at time* T *if and only if there is a constant* $C > 0$ *such that for all* $(z, \hat{z}) \in \mathcal{H}_5$ *it holds that*

$$|(z_0, \hat{z}_0)|^2_{H_0^2(0,1)\times L^2(0,1)} \le C\mathbb{E}\int_0^T \Big[z_{xx}^2(0,t) + z_{xxx}^2(0,t)\Big]dt \ \ \forall\ (z_0, \hat{z}_0) \in \mathcal{H}_1. \tag{4.4}$$

The proof of Proposition 4.6 is based on a duality argument and can be found in [17]. From Proposition 4.6, the null controllability result can be reduced to an observability estimate. Now we give an observability estimate for (4.3) by means of the Carleman estimate (2.2).

THEOREM 4.7. *There is a constant* $C > 0$*, such that all solutions* $(z, \hat{z}) \in \mathcal{H}_5$ *to* (4.3) *satisfy* (4.4)*.*

Proof. By (4.3) and Theorem 2.1, there exist $\mu_1 > 0$ and $\lambda_1 = \lambda_1(\mu)$, such that for all $\mu \ge \mu_1$ and $\lambda \ge \lambda_1(\mu)$,

$$\begin{aligned} &\mathbb{E}\int_Q \theta_1^2\Big[\lambda^7\mu^8\varphi_1^7 z^2 + \lambda^5\mu^6\varphi_1^5 z_x^2 + \lambda^3\mu^4\varphi_1^3(z_{xx}^2 + \hat{z}^2) + \lambda\mu^2\varphi_1(z_{xxx}^2 + \hat{z}_x^2)\Big]\mathrm{d}x\mathrm{d}t \\ &\le C\mathbb{E}\int_0^T \theta_1^2\Big[\lambda\mu\varphi_1 z_{xxx}^2(0,t) + \lambda^3\mu^3\varphi_1^3 z_{xx}^2(0,t)\Big]\mathrm{d}t. \end{aligned} \tag{4.5}$$

Set

$$\mathcal{E}(t) = \frac{1}{2}\mathbb{E}\int_0^1 \Big[z^2(x,t) + \hat{z}^2(x,t) + z_x^2(x,t) + z_{xx}^2(x,t)\Big]\mathrm{d}x,\ \ t\in[0,T].$$

Then, by Itô's formula, for any s,t satisfying $0\le s\le t\le T$, we have

$$\begin{aligned}
&\mathcal{E}(t)-\mathcal{E}(s)\\
&= \frac{1}{2}\mathbb{E}\int_s^t\int_0^1 \Big[2z\mathrm{d}z + (\mathrm{d}z)^2\Big]\mathrm{d}x + \frac{1}{2}\mathbb{E}\int_s^t\int_0^1 \Big[2\hat{z}\mathrm{d}\hat{z} + \big(\mathrm{d}\hat{z}\big)^2\Big]\mathrm{d}x\\
&\quad + \frac{1}{2}\mathbb{E}\int_s^t\int_0^1 \Big[2z_x\mathrm{d}z_x + (\mathrm{d}z_x)^2\Big]\mathrm{d}x + \frac{1}{2}\mathbb{E}\int_s^t\int_0^1 \Big[2z_{xx}\mathrm{d}z_{xx} + (\mathrm{d}z_{xx})^2\Big]\mathrm{d}x\\
&= \frac{1}{2}\mathbb{E}\int_s^t\int_0^1 \Big\{2z\hat{z} + \big(a_3 z\big)^2 + 2z_x\hat{z}_x + \Big[\big(a_3z\big)_x\Big]^2 + 2z_{xx}\hat{z}_{xx} + 2\Big[\big(a_3z\big)_{xx}\Big]^2\\
&\qquad\qquad + \hat{z}\Big[-z_{xxxx} + \big(a_1 - a_{2,x}\big)z - a_2 z_x\Big]\Big\}\mathrm{d}x\mathrm{d}\tau\\
&= \frac{1}{2}\mathbb{E}\int_s^t\int_0^1 \Big\{2z\hat{z} + \big(a_3 z\big)^2 - 2z_{xx}\hat{z} + \Big[\big(a_3z\big)_x\Big]^2 + 2\Big[\big(a_3z\big)_{xx}\Big]^2\\
&\qquad\qquad + \hat{z}\Big[\big(a_1 - a_{2,x}\big)z - a_2 z_x\Big]\Big\}\mathrm{d}x\mathrm{d}\tau,
\end{aligned}$$

and hence

$$\mathcal{E}(s) \le \mathcal{E}(t) + C\mathbb{E}\int_s^t\int_0^1 \big(z^2 + \hat{z}^2 + z_x^2 + z_{xx}^2\big)\mathrm{d}x\mathrm{d}\tau.$$

By Gronwall's inequality, one can obtain that

$$\mathcal{E}(s) \le C\mathcal{E}(t). \tag{4.6}$$

Taking $\lambda=\lambda_1, \mu=\mu_1$ in (4.5), and recalling the definitions of θ_1,φ_1, we have

$$\mathbb{E}\int_{\frac{T}{4}}^{\frac{3T}{4}}\int_0^1 \Big[z^2 + z_x^2 + \big(z_{xx}^2 + \hat{z}^2\big)\Big]\mathrm{d}x\mathrm{d}t \le C\mathbb{E}\int_0^T \Big[z_{xxx}^2(0,t) + z_{xx}^2(0,t)\Big]\mathrm{d}t. \tag{4.7}$$

By (4.6) and (4.7), we complete the proof of Theorem 4.7. ∎

Proof of Theorem 2.7. It follows from Proposition 4.5, Proposition 4.6, and Theorem 4.7 immediately. ∎

Although it is necessary to put controls f and g on the whole domain, one may suspect that Theorem 4.2 is trivial. However, we have the following negative result.

THEOREM 4.8. *The system* (2.4) *is not exactly controllable at any time* $T>0$ *provided that one of the following three conditions is satisfied:*
(1) $\mathrm{supp} f \subseteq G_2$*, where* $G_2 \subsetneqq (0,1)$;
(2) $\mathrm{supp} g \subseteq G_2$*, where* $G_2 \subsetneqq (0,1)$;
(3) $h_1 = h_2 = 0$.

From the above negative result, we find that none of the two internal controls f,g and boundary controls can be ignored, and internal controls must be effective everywhere in the domain $(0,1)$. The proof of Theorem 4.8 is based on the contradiction argument and the known conclusions [22, Lemma 2.1]. Since it is similar to the proof of Theorem 2.3 in [17], we omit it here.

Remark 4.9. Although boundary controls h_1, h_2 cannot be dropped simultaneously in (2.4), it is worth studying whether one of them can be removed. It seems to be true from the existing controllability results of deterministic beam systems.

In this paper, we prove that (2.4) is exactly controllable, but (4.1) is not exactly controllable at any time. Therefore, from the viewpoint of control theory, the refined system (2.4) is a more reasonable model than the classical system (4.1).

5. Inverse problem. In this section, we are devoted to proving the inverse source result of the refined stochastic beam equation through the Carleman estimate (2.3).

Proof of Theorem 2.12. Let $(y, \hat{y})$ be the solution of (2.5). Set $y = Rh$, $\hat{y} = R\hat{h}$. Then it can be seen that $(h, \hat{h})$ satisfies that

$$
(5.1)\quad \begin{cases}
\mathrm{d}h = \Big[\hat{h} + A(t) - \dfrac{R_t}{R}h\Big]\mathrm{d}t + \Big[c_1 h + B(t)\Big]\mathrm{d}W(t) & \text{in } Q,\\
\mathrm{d}\hat{h} + h_{xxxx}\mathrm{d}t = \Big[c_2 h + c_3 h_x + c_3\dfrac{R_x}{R}h - \dfrac{R_t}{R}\hat{h} - \dfrac{4R_x}{R}h_{xxx} & \\
\qquad - \dfrac{6R_{xx}}{R}h_{xx} - \dfrac{4R_{xxx}}{R}h_x - \dfrac{R_{xxxx}}{R}h + H(t)\Big]\mathrm{d}t & \\
\qquad + \Big[c_4 h + P(t)\Big]\mathrm{d}W(t) & \text{in } Q,\\
h(0,t) = h_x(0,t) = 0 & \text{on } (0,T),\\
h(1,t) = h_x(1,t) = 0 & \text{on } (0,T),\\
h(x,0) = h_0(x) = \dfrac{y_0(x)}{R(x,0)}, \quad \hat{h}(x,0) = \hat{h}_0(x) = \dfrac{\hat{y}_0(x)}{R(x,0)} & \text{in } (0,1).
\end{cases}
$$

Put $u = h_x$, $\hat{u} = \hat{h}_x$. Then, by (2.6) and (5.1), $(u, \hat{u})$ solves

$$
(5.2)\quad \begin{cases}
\mathrm{d}u = \Big[\hat{u} - \dfrac{R_t}{R}u - \Big(\dfrac{R_t}{R}\Big)_x h\Big]\mathrm{d}t + \big(c_1 u + c_{1,x}h\big)\mathrm{d}W(t) & \text{in } Q,\\
\mathrm{d}\hat{u} + u_{xxxx}\mathrm{d}t = \Big[\widetilde{E_1}(u,\hat{u}) + \widetilde{F_1}(h,\hat{h})\Big]\mathrm{d}t & \\
\qquad + \big(c_4 u + c_{4,x}h\big)\mathrm{d}W(t) & \text{in } Q,\\
u(0,t) = u_x(0,t) = 0 & \text{on } (0,T),\\
u(1,t) = u_x(1,t) = 0 & \text{on } (0,T),\\
u(x,0) = h_{0,x}(x), \quad \hat{u}(x,0) = \hat{h}_{0,x}(x) & \text{in } (0,1),
\end{cases}
$$

where

$$
\begin{aligned}
\widetilde{E_1}(u,\hat{u}) = {}& c_2 u + c_3 u_x + c_3\frac{R_x}{R}u - \frac{R_t}{R}\hat{u} - \frac{4R_x}{R}u_{xxx} - \frac{6R_{xx}}{R}u_{xx} - \frac{4R_{xxx}}{R}u_x\\
& - \frac{R_{xxxx}}{R}u + c_{3,x}u - \Big(\frac{4R_x}{R}\Big)_x u_{xx} - \Big(\frac{6R_{xx}}{R}\Big)_x u_x - \Big(\frac{4R_{xxx}}{R}\Big)_x u,
\end{aligned}
$$

and

$$
\widetilde{F_1}(h,\hat{h}) = c_{2,x}h + \Big(c_3\frac{R_x}{R}\Big)_x h - \Big(\frac{R_t}{R}\Big)_x \hat{h} - \Big(\frac{R_{xxxx}}{R}\Big)_x h.
$$

By means of $y(0,t)=\hat{y}(0,t)=0$ in $(0,T)$, we have

$$
\begin{aligned}
h &= \int_0^x h_x(\eta,t)\mathrm{d}\eta = \int_0^x u(\eta,t)\mathrm{d}\eta,\\
\hat{h} &= \int_0^x \hat{h}_x(\eta,t)\mathrm{d}\eta = \int_0^x \hat{u}(\eta,t)\mathrm{d}\eta.
\end{aligned}
$$

Therefore, together with (5.2), $(u,\hat{u})$ satisfies

$$
\text{(5.3)}\qquad
\begin{cases}
\mathrm{d}u = \left[\hat{u} - \dfrac{R_t}{R}u - \left(\dfrac{R_t}{R}\right)_x \displaystyle\int_0^x u(\eta,t)\mathrm{d}\eta\right]\mathrm{d}t\\
\qquad + \left[c_1 u + c_{1,x}\displaystyle\int_0^x u(\eta,t)\mathrm{d}\eta\right]\mathrm{d}W(t) & \text{in } Q,\\
\mathrm{d}\hat{u} + u_{xxxx}\mathrm{d}t = \left[\widetilde{E_1}(u,\hat{u}) + \widetilde{F_2}(u,\hat{u})\right]\mathrm{d}t\\
\qquad + \left[c_4 u + c_{4,x}\displaystyle\int_0^x u(\eta,t)\mathrm{d}\eta\right]\mathrm{d}W(t) & \text{in } Q,\\
u(0,t) = u_x(0,t) = 0 & \text{on } (0,T),\\
u(1,t) = u_x(1,t) = 0 & \text{on } (0,T),\\
u(x,0) = h_{0,x}(x), \quad \hat{u}(x,0) = \hat{h}_{0,x}(x) & \text{in } (0,1),
\end{cases}
$$

where

$$
\widetilde{F_2}(u,\hat{u}) = \left[c_{2,x} + \left(c_3\frac{R_x}{R}\right)_x - \left(\frac{R_{xxxx}}{R}\right)_x\right]\int_0^x u(\eta,t)\mathrm{d}\eta - \left(\frac{R_t}{R}\right)_x\int_0^x \hat{u}(\eta,t)\mathrm{d}\eta.
$$

From (5.3), we know that $u(0,t)=u(1,t)=u_x(0,t)=u_x(1,t)=0$ in $(0,T)$. Applying the Carleman estimate (2.3) to (5.3), there exist $\mu_1\geq 1$ and $\lambda_1=\lambda_1(\mu)\geq 1$, such that for all $\mu\geq\mu_1$ and $\lambda\geq\lambda_1(\mu)$,

(5.4)

$$
\begin{aligned}
&\mathbb{E}\int_Q \theta_2^2\left[\lambda^7\mu^8\varphi_2^7u^2 + \lambda^5\mu^6\varphi_2^5u_x^2 + \lambda^3\mu^4\varphi_2^3\left(u_{xx}^2+\hat{u}^2\right) + \lambda\mu^2\varphi_2\left(u_{xxx}^2+\hat{u}_x^2\right)\right]\mathrm{d}x\mathrm{d}t\\
&\leq C\mathbb{E}\int_Q \theta_2^2\left[\lambda^6\mu^6\varphi_2^6\left(\int_0^x u(\eta,t)\mathrm{d}\eta\right)^2 + \lambda^2\mu^8\varphi_2^2\left(\int_0^x u(\eta,t)\mathrm{d}\eta\right)^2\right]\mathrm{d}x\mathrm{d}t\\
&\quad + C\mathbb{E}\int_0^T\int_{G_0}\theta_2^2\left(\lambda^7\mu^8\varphi_2^7u^2 + \lambda^5\mu^6\varphi_2^5u_x^2 + \lambda^3\mu^4\varphi_2^3u_{xx}^2 + \lambda\mu^2\varphi_2u_{xxx}^2\right)\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Since

$$
\left|\int_0^x u(\eta,t)\mathrm{d}\eta\right|^2 \leq \int_0^1 |u(\eta,t)|^2\mathrm{d}\eta,
$$

we see that

$$
\int_Q\left|\int_0^x u(\eta,t)\mathrm{d}\eta\right|^2\mathrm{d}x\mathrm{d}t \leq \int_Q |u(\eta,t)|^2\mathrm{d}x\mathrm{d}t.
$$

Combining the above inequality with (5.4), there exist $\mu_2>0$ and $\lambda_2=\lambda_2(\mu)$, such that for all $\mu\geq\mu_2$ and $\lambda\geq\lambda_2(\mu)$,

$$
\begin{aligned}
&\mathbb{E}\int_Q \theta_2^2\left[\lambda^7\mu^8\varphi_2^7u^2 + \lambda^5\mu^6\varphi_2^5u_x^2 + \lambda^3\mu^4\varphi_2^3\left(u_{xx}^2+\hat{u}^2\right) + \lambda\mu^2\varphi_2\left(u_{xxx}^2+\hat{u}_x^2\right)\right]\mathrm{d}x\mathrm{d}t\\
&\leq C\mathbb{E}\int_0^T\int_{G_0}\theta_2^2\left(\lambda^7\mu^8\varphi_2^7u^2 + \lambda^5\mu^6\varphi_2^5u_x^2 + \lambda^3\mu^4\varphi_2^3u_{xx}^2 + \lambda\mu^2\varphi_2u_{xxx}^2\right)\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Together the above inequality with $y = 0$ in $G_0 \times (0, T)$ implies that

$$u = 0 \text{ in } G_0 \times (0, T), \ \mathbb{P}\text{-a.s.},$$

that we have

$$u = \hat{u} = 0 \text{ in } Q, \ \mathbb{P}\text{-a.s.},$$

and hence that

$$y = \hat{y} = 0 \text{ in } Q, \ \mathbb{P}\text{-a.s.},$$

which means

$$A(\cdot) = B(\cdot) = H(\cdot) = P(\cdot) = 0 \text{ in } (0, T), \ \mathbb{P}\text{-a.s.} \qquad \square$$

6. Summary.

This paper considers the global Carleman estimates of refined stochastic beam equations with the uncertainty between the velocity and displacement of the beam model. By establishing a fundamental weighted identity and properly choosing weight functions, two Carleman estimates are established, with which the exact controllability of the refined system is proved and, meanwhile, the uniqueness of an inverse drift source problem is obtained without any requirement on the initial and terminal data.

It will be interesting to further consider the following problems:

(i) whether one of the boundary controls can be removed in the controllability result;

(ii) whether the refined system is exactly controllable when the control $g \in L^2_{\mathbb{F}}(0, T; L^2(0, 1))$;

(iii) how to get the quantitative estimates for the inverse diffusion source problem;

(iv) how to obtain the inverse source result by boundary observation or internal observation alone.

Appendix A. Proof of Lemma 3.1.

By (3.1) and $z = \theta y$, $\hat{z} = \theta \hat{y} + \ell_t z$, we obtain that

$$\mathrm{d}z = \mathrm{d}\big(\theta y\big) = \theta \ell_t y \mathrm{d}t + \theta \mathrm{d}y = \hat{z}\mathrm{d}t + \theta f_1 \mathrm{d}t + \theta g_1 \mathrm{d}W(t). \tag{A.1}$$

Hence,

$$\begin{aligned} \mathrm{d}\hat{y} &= \mathrm{d}\Big[\theta^{-1}\big(\hat{z} - \ell_t z\big)\Big] = \theta^{-1}\Big[\mathrm{d}\hat{z} - \ell_{tt} z \mathrm{d}t - \ell_t \mathrm{d}z - \ell_t\big(\hat{z} - \ell_t z\big)\mathrm{d}t\Big] \\ &= \theta^{-1}\Big[\mathrm{d}\hat{z} - 2\ell_t \hat{z}\mathrm{d}t + \big(\ell_t^2 - \ell_{tt}\big) z \mathrm{d}t - \theta \ell_t f_1 \mathrm{d}t - \theta \ell_t g_1 \mathrm{d}W(t)\Big]. \end{aligned}$$

Recalling $y = \theta^{-1} z$, it holds that

$$\begin{aligned} y_{xxxx} = \theta^{-1}\Big[& z_{xxxx} - 4\ell_x z_{xxx} + 6\big(\ell_x^2 - \ell_{xx}\big) z_{xx} - 4D z_x \\ &+ \big(\ell_x^4 - 6\ell_x^2 \ell_{xx} + 3\ell_{xx}^2 + 4\ell_x \ell_{xxx} - \ell_{xxxx}\big) z\Big]. \end{aligned}$$

Therefore,

$$\begin{aligned} &\theta\big(\mathrm{d}\hat{y} + y_{xxxx}\mathrm{d}t\big) \\ &= \mathrm{d}\hat{z} + z_{xxxx}\mathrm{d}t - 2\ell_t \hat{z}\mathrm{d}t - 4\ell_x z_{xxx}\mathrm{d}t + 6\big(\ell_x^2 - \ell_{xx}\big) z_{xx}\mathrm{d}t \\ &\quad - 4D z_x \mathrm{d}t + \big(\ell_x^4 - 6\ell_x^2 \ell_{xx} + 3\ell_{xx}^2 + 4\ell_x \ell_{xxx} - \ell_{xxxx} + \ell_t^2 - \ell_{tt}\big) z \mathrm{d}t \\ &\quad - \theta \ell_t f_1 \mathrm{d}t - \theta \ell_t g_1 \mathrm{d}W(t). \end{aligned}$$

From the definitions of Φ and Ψ, one can get that

$$\begin{aligned}
&\theta I\big(\mathrm{d}\hat{y} + y_{xxxx}\mathrm{d}t\big)\\
&= I\Big[\mathrm{d}\hat{z} + z_{xxxx}\mathrm{d}t + 6\big(\ell_x^2 - \ell_{xx}\big)z_{xx}\mathrm{d}t + \big(\Phi - \Psi\big)z\mathrm{d}t\Big]\\
&\qquad + I^2 - \theta I\ell_t f_1\mathrm{d}t - \theta I\ell_t g_1\mathrm{d}W(t)\\
&= I_1 + I_2 + I_3 + I_4 + I^2 - \theta I\ell_t f_1\mathrm{d}t - \theta I\ell_t g_1\mathrm{d}W(t),
\end{aligned} \tag{A.2}$$

where $I_1 = I\mathrm{d}\hat{z}, I_2 = Iz_{xxxx}\mathrm{d}t, I_3 = 6I\big(\ell_x^2 - \ell_{xx}\big)z_{xx}\mathrm{d}t, I_4 = I\big(\Phi - \Psi\big)z\mathrm{d}t$. Now we analyze the right-hand side of (A.2). For the first one, we have

$$\begin{aligned}
I_1 &= \mathrm{d}\big(-\ell_t\hat{z}^2 - 4\ell_x z_{xxx}\hat{z} - 4Dz_x\hat{z} + \Psi z\hat{z}\big) + \ell_{tt}\hat{z}^2\mathrm{d}t + \ell_t\big(\mathrm{d}\hat{z}\big)^2\\
&\quad + 4\ell_{xt}z_{xxx}\hat{z}\mathrm{d}t + 4\ell_x\hat{z}\mathrm{d}z_{xxx} + 4\ell_x\mathrm{d}\hat{z}\mathrm{d}z_{xxx} + 4D_t z_x\hat{z}\mathrm{d}t\\
&\quad + 4D\hat{z}\mathrm{d}z_x + 4D\mathrm{d}\hat{z}\mathrm{d}z_x - \Psi_t z\hat{z}\mathrm{d}t - \Psi\hat{z}\mathrm{d}z - \Psi\mathrm{d}\hat{z}\mathrm{d}z.
\end{aligned}$$

Because of $\mathrm{d}z = \hat{z}\mathrm{d}t + \theta f_1\mathrm{d}t + \theta g_1\mathrm{d}W(t)$, it follows that

$$\begin{aligned}
&4\ell_x\hat{z}\mathrm{d}z_{xxx} + 4D\hat{z}\mathrm{d}z_x - \Psi\hat{z}\mathrm{d}z\\
&= 4\big(\ell_x\hat{z}\mathrm{d}z_{xx}\big)_x - 4\big(\ell_{xx}\hat{z} + \ell_x\hat{z}_x\big)\Big[\hat{z}\mathrm{d}t + \theta f_1\mathrm{d}t + \theta g_1\mathrm{d}W(t)\Big]_{xx}\\
&\quad + 4D\hat{z}\Big[\hat{z}\mathrm{d}t + \theta f_1\mathrm{d}t + \theta g_1\mathrm{d}W(t)\Big]_x - \Psi\hat{z}\Big[\hat{z}\mathrm{d}t + \theta f_1\mathrm{d}t + \theta g_1\mathrm{d}W(t)\Big]\\
&= 4\big(\ell_x\hat{z}\mathrm{d}z_{xx}\big)_x - \big(4\ell_{xx}\hat{z}\hat{z}_x + 2\ell_x\hat{z}_x^2 - 2D\hat{z}^2\big)_x\mathrm{d}t + 4\ell_{xx}\hat{z}_x^2\mathrm{d}t + 4\ell_{xxx}\hat{z}\hat{z}_x\mathrm{d}t\\
&\quad + 2\ell_{xx}\hat{z}_x^2\mathrm{d}t - 2D_x\hat{z}^2\mathrm{d}t - 4\big(\ell_{xx}\hat{z} + \ell_x\hat{z}_x\big)\big(\theta f_1\big)_{xx}\mathrm{d}t + 4D\hat{z}\big(\theta f_1\big)_x\mathrm{d}t - \Psi\hat{z}^2\mathrm{d}t\\
&\quad - \Psi\hat{z}\theta f_1\mathrm{d}t - \Big[4\big(\ell_{xx}\hat{z} + \ell_x\hat{z}_x\big)\big(\theta g_1\big)_{xx} - 4D\hat{z}\big(\theta g_1\big)_x + \Psi\hat{z}\theta g_1\Big]\mathrm{d}W(t)\\
&= 4\big(\ell_x\hat{z}\mathrm{d}z_{xx}\big)_x - 2\big(2\ell_{xx}\hat{z}\hat{z}_x + \ell_x\hat{z}_x^2 - D\hat{z}^2 - \ell_{xxx}\hat{z}^2\big)_x\mathrm{d}t + 6\ell_{xx}\hat{z}_x^2\mathrm{d}t\\
&\quad - \big(2\ell_{xxxx} + 2D_x + \Psi\big)\hat{z}^2\mathrm{d}t - 4\big(\ell_{xx}\hat{z} + \ell_x\hat{z}_x\big)\big(\theta f_1\big)_{xx}\mathrm{d}t + 4D\hat{z}\big(\theta f_1\big)_x\mathrm{d}t\\
&\quad - \Psi\hat{z}\theta f_1\mathrm{d}t - \Big[4\big(\ell_{xx}\hat{z} + \ell_x\hat{z}_x\big)\big(\theta g_1\big)_{xx} - 4D\hat{z}\big(\theta g_1\big)_x + \Psi\hat{z}\theta g_1\Big]\mathrm{d}W(t).
\end{aligned}$$

So,

$$\begin{aligned}
I_1 &= \mathrm{d}\big(-\ell_t\hat{z}^2 - 4\ell_x z_{xxx}\hat{z} - 4Dz_x\hat{z} + \Psi z\hat{z}\big) + 4\big(\ell_x\hat{z}\mathrm{d}z_{xx}\big)_x\\
&\quad - \big(4\ell_{xx}\hat{z}\hat{z}_x + 2\ell_x\hat{z}_x^2 - 2D\hat{z}^2 - 2\ell_{xxx}\hat{z}^2\big)_x\mathrm{d}t + 6\ell_{xx}\hat{z}_x^2\mathrm{d}t - \Psi_t\hat{z}z\mathrm{d}t\\
&\quad + \big(\ell_{tt} - 2\ell_{xxxx} - 2D_x - \Psi\big)\hat{z}^2\mathrm{d}t + 4D_t\hat{z}z_x\mathrm{d}t + 4\ell_{xt}z_{xxx}\hat{z}\mathrm{d}t\\
&\quad - 4\big(\ell_{xx}\hat{z} + \ell_x\hat{z}_x\big)\big(\theta f_1\big)_{xx}\mathrm{d}t + 4D\hat{z}\big(\theta f_1\big)_x\mathrm{d}t - \Psi\hat{z}\theta f_1\mathrm{d}t\\
&\quad + \ell_t\big(\mathrm{d}\hat{z}\big)^2 + 4\ell_x\mathrm{d}\hat{z}\mathrm{d}z_{xxx} + 4D\mathrm{d}\hat{z}\mathrm{d}z_x - \Psi\mathrm{d}z\mathrm{d}\hat{z}\\
&\quad - \Big[4\big(\ell_{xx}\hat{z} + \ell_x\hat{z}_x\big)\big(\theta g_1\big)_{xx} - 4D\hat{z}\big(\theta g_1\big)_x + \Psi\hat{z}\theta g_1\Big]\mathrm{d}W(t).
\end{aligned} \tag{A.3}$$

For the second one, we have

$$\begin{aligned}
I_2 &= \big(-2\ell_t\hat{z}z_{xxx} - 2\ell_x z_{xxx}^2 - 4Dz_x z_{xxx} + \Psi z z_{xxx}\big)_x\mathrm{d}t + 2\ell_{tx}\hat{z}z_{xxx}\mathrm{d}t\\
&\quad + 2\ell_t\hat{z}_x z_{xxx}\mathrm{d}t + 2\ell_{xx}z_{xxx}^2\mathrm{d}t - \Psi_x z z_{xxx}\mathrm{d}t - \Psi z_x z_{xxx}\mathrm{d}t\\
&\quad + 4Dz_{xx}z_{xxx}\mathrm{d}t + 4D_x z_x z_{xxx}\mathrm{d}t\\
&= \big(-2\ell_t\hat{z}z_{xxx} - 2\ell_x z_{xxx}^2 - 4Dz_x z_{xxx} + \Psi z z_{xxx} + 2\ell_t\hat{z}_x z_{xx}\\
&\quad - \Psi z_x z_{xx} + 2Dz_{xx}^2 + 4D_x z_x z_{xx}\big)_x\mathrm{d}t + 2\ell_{tx}\hat{z}z_{xxx}\mathrm{d}t - 2\ell_t\hat{z}_{xx}z_{xx}\mathrm{d}t\\
&\quad - 2\ell_{tx}\hat{z}_x z_{xx}\mathrm{d}t + 2\ell_{xx}z_{xxx}^2\mathrm{d}t - \Psi_x z z_{xxx}\mathrm{d}t + \Psi z_{xx}^2\mathrm{d}t + \Psi_x z_x z_{xx}\mathrm{d}t\\
&\quad - 6D_x z_{xx}^2\mathrm{d}t - 4D_{xx}z_x z_{xx}\mathrm{d}t.
\end{aligned} \tag{A.4}$$

By (A.1), it follows that

$$
\begin{aligned}
&2\ell_t \hat{z}_{xx} z_{xx} \mathrm{d}t = 2\ell_t z_{xx}\Big[\mathrm{d}z - \theta f_1 \mathrm{d}t - \theta g_1 \mathrm{d}W(t)\Big]_{xx} \\
&= 2\ell_t z_{xx}\mathrm{d}z_{xx} - 2\ell_t z_{xx}\big(\theta f_1\big)_{xx}\mathrm{d}t - 2\ell_t z_{xx}\big(\theta g_1\big)_{xx}\mathrm{d}W(t) \\
&= \mathrm{d}\big(\ell_t z_{xx}^2\big) - \ell_{tt} z_{xx}^2 \mathrm{d}t - \ell_t\big(\mathrm{d}z_{xx}\big)^2 - 2\ell_t z_{xx}\big(\theta f_1\big)_{xx}\mathrm{d}t - 2\ell_t z_{xx}\big(\theta g_1\big)_{xx}\mathrm{d}W(t).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
I_2 = \big(&-2\ell_t \hat{z} z_{xxx} - 2\ell_x z_{xxx}^2 - 4D z_x z_{xxx} + \Psi z z_{xxx} + 2\ell_t \hat{z}_x z_{xx} \\
&- \Psi z_x z_{xx} + 2D z_{xx}^2 + 4D_x z_x z_{xx}\big)_x \mathrm{d}t + 2\ell_{tx}\hat{z} z_{xxx}\mathrm{d}t - 2\ell_{tx}\hat{z}_x z_{xx}\mathrm{d}t \\
&+ 2\ell_{xx} z_{xxx}^2 \mathrm{d}t - \Psi_x z z_{xxx}\mathrm{d}t + \big(\Psi - 6D_x + \ell_{tt}\big) z_{xx}^2 \mathrm{d}t \\
&+ \big(\Psi_x - 4D_{xx}\big) z_x z_{xx}\mathrm{d}t - \mathrm{d}\big(\ell_t z_{xx}^2\big) + \ell_t\big(\mathrm{d}z_{xx}\big)^2 \\
&+ 2\ell_t z_{xx}\big(\theta f_1\big)_{xx}\mathrm{d}t + 2\ell_t z_{xx}\big(\theta g_1\big)_{xx}\mathrm{d}W(t).
\end{aligned} \tag{A.5}
$$

For the third one, we have

$$
\begin{aligned}
I_3 = &-6\Big[2\big(\ell_t\ell_x^2 - \ell_t\ell_{xx}\big)\hat{z} z_x + 2\big(\ell_x^3 - \ell_x\ell_{xx}\big) z_{xx}^2 - \big(\Psi\ell_x^2 - \Psi\ell_{xx}\big) z z_x \\
&\quad + 2D\big(\ell_x^2 - \ell_{xx}\big) z_x^2\Big]_x \mathrm{d}t + 12\big(\ell_t\ell_x^2 - \ell_t\ell_{xx}\big)_x \hat{z} z_x \mathrm{d}t + 12\big(\ell_t\ell_x^2 - \ell_t\ell_{xx}\big)\hat{z}_x z_x \mathrm{d}t \\
&\quad + 12\big(\ell_x^3 - \ell_x\ell_{xx}\big)_x z_{xx}^2 \mathrm{d}t - 6\big(\Psi\ell_x^2 - \Psi\ell_{xx}\big) z_x^2 \mathrm{d}t + 12\Big[D\big(\ell_x^2 - \ell_{xx}\big)\Big]_x z_x^2 \mathrm{d}t \\
&\quad - 6\big(\Psi\ell_x^2 - \Psi\ell_{xx}\big)_x z z_x \mathrm{d}t \\
= &-3\Big[4\big(\ell_t\ell_x^2 - \ell_t\ell_{xx}\big)\hat{z} z_x + 4\big(\ell_x^3 - \ell_x\ell_{xx}\big) z_{xx}^2 - 2\big(\Psi\ell_x^2 - \Psi\ell_{xx}\big) z z_x \\
&+ 4D\big(\ell_x^2 - \ell_{xx}\big) z_x^2 + \big(\Psi\ell_x^2 - \Psi\ell_{xx}\big)_x z^2\Big]_x \mathrm{d}t + 12\big(\ell_t\ell_x^2 - \ell_t\ell_{xx}\big)_x \hat{z} z_x \mathrm{d}t \\
&+ 12\big(\ell_t\ell_x^2 - \ell_t\ell_{xx}\big) z_x \hat{z}_x \mathrm{d}t + 12(\ell_x^3 - \ell_x\ell_{xx})_x z_{xx}^2 \mathrm{d}t + 3\big(\Psi\ell_x^2 - \Psi\ell_{xx}\big)_{xx} z^2 \mathrm{d}t \\
&+ 12\Big[D\big(\ell_x^2 - \ell_{xx}\big)\Big]_x z_x^2 \mathrm{d}t - 6\big(\Psi\ell_x^2 - \Psi\ell_{xx}\big) z_x^2 \mathrm{d}t.
\end{aligned}
$$

By (A.1), it follows that

$$
\begin{aligned}
&12\big(\ell_t\ell_x^2 - \ell_t\ell_{xx}\big) z_x \hat{z}_x \mathrm{d}t \\
&= 12\big(\ell_t\ell_x^2 - \ell_t\ell_{xx}\big) z_x\Big[\mathrm{d}z - \theta f_1 \mathrm{d}t - \theta g_1 \mathrm{d}W(t)\Big]_x \\
&= 12\big(\ell_t\ell_x^2 - \ell_t\ell_{xx}\big) z_x \mathrm{d}z_x - 12\big(\ell_t\ell_x^2 - \ell_t\ell_{xx}\big) z_x\big(\theta f_1\big)_x \mathrm{d}t \\
&\quad -12\big(\ell_t\ell_x^2 - \ell_t\ell_{xx}\big) z_x\big(\theta g_1\big)_x \mathrm{d}W(t) \\
&= 6\mathrm{d}\Big[\big(\ell_t\ell_x^2 - \ell_t\ell_{xx}\big) z_x^2\Big] - 6\big(\ell_t\ell_x^2 - \ell_t\ell_{xx}\big)_t z_x^2 \mathrm{d}t - 6\big(\ell_t\ell_x^2 - \ell_t\ell_{xx}\big)\big(\mathrm{d}z_x\big)^2 \\
&\quad -12\big(\ell_t\ell_x^2 - \ell_t\ell_{xx}\big) z_x\big(\theta f_1\big)_x \mathrm{d}t - 12\big(\ell_t\ell_x^2 - \ell_t\ell_{xx}\big) z_x\big(\theta g_1\big)_x \mathrm{d}W(t).
\end{aligned}
$$

Thus, one can get

(A.6)

$$
\begin{aligned}
I_3 = & -3\Big[4\big(\ell_t\ell_x^2-\ell_t\ell_{xx}\big)\hat{z}z_x+4\big(\ell_x^3-\ell_x\ell_{xx}\big)z_{xx}^2-2\big(\Psi\ell_x^2-\Psi\ell_{xx}\big)zz_x\\
& +4D\big(\ell_x^2-\ell_{xx}\big)z_x^2+\big(\Psi\ell_x^2-\Psi\ell_{xx}\big)_x z^2\Big]_x\mathrm{d}t+6\mathrm{d}\Big[(\ell_t\ell_x^2-\ell_t\ell_{xx})z_x^2\Big]\\
& +6\Big[2\big(D\big(\ell_x^2-\ell_{xx}\big)\big)_x-\big(\Psi\ell_x^2-\Psi\ell_{xx}\big)-\big(\ell_t\ell_x^2-\ell_t\ell_{xx}\big)_t\Big]z_x^2\mathrm{d}t\\
& +12\big(\ell_x^3-\ell_x\ell_{xx}\big)_x z_{xx}^2\mathrm{d}t+3\big(\Psi\ell_x^2-\Psi\ell_{xx}\big)_{xx}z^2\mathrm{d}t+12\big(\ell_t\ell_x^2-\ell_t\ell_{xx}\big)_x\hat{z}z_x\mathrm{d}t\\
& -6\big(\ell_t\ell_x^2-\ell_t\ell_{xx}\big)\big(\mathrm{d}z_x\big)^2-12\big(\ell_t\ell_x^2-\ell_t\ell_{xx}\big)z_x\big(\theta f_1\big)_x\mathrm{d}t\\
& -12\big(\ell_t\ell_x^2-\ell_t\ell_{xx}\big)z_x\big(\theta g_1\big)_x\mathrm{d}W(t).
\end{aligned}
$$

For the fourth one, we have

$$
\begin{aligned}
I_4 = & -2\ell_t\big(\Phi-\Psi\big)z\Big[\mathrm{d}z-\theta f_1\mathrm{d}t-\theta g_1\mathrm{d}W(t)\Big]+\Psi\big(\Phi-\Psi\big)z^2\mathrm{d}t\\
& -2\Big[2\ell_x\big(\Phi-\Psi\big)zz_{xx}+D\big(\Phi-\Psi\big)z^2\Big]_x\mathrm{d}t+4\big(\ell_x\Phi-\ell_x\Psi\big)_x zz_{xx}\mathrm{d}t\\
& +4\big(\ell_x\Phi-\ell_x\Psi\big)z_xz_{xx}\mathrm{d}t+2\Big[D\big(\Phi-\Psi\big)\Big]_x z^2\mathrm{d}t\\
= & \,\mathrm{d}\Big[-\ell_t\big(\Phi-\Psi\big)z^2\Big]-2\Big[2\ell_x\big(\Phi-\Psi\big)zz_{xx}+D\big(\Phi-\Psi\big)z^2-2\big(\ell_x\Phi-\ell_x\Psi\big)_x zz_x\\
& -\big(\ell_x\Phi-\ell_x\Psi\big)z_x^2+\big(\ell_x\Phi-\ell_x\Psi\big)_{xx}z^2\Big]_x\mathrm{d}t+\Big[\big(\ell_t\Phi-\ell_t\Psi\big)_t+\Psi\big(\Phi-\Psi\big)\\
& +2\big(\ell_x\Phi-\ell_x\Psi\big)_{xxx}+2\big(D\Phi-D\Psi\big)_x\Big]z^2\mathrm{d}t+2\ell_t\big(\Phi-\Psi\big)z\theta f_1\mathrm{d}t\\
& -6\big(\ell_x\Phi-\ell_x\Psi\big)_x z_x^2\mathrm{d}t+\big(\ell_t\Phi-\ell_t\Psi\big)\big(\mathrm{d}z\big)^2+2\ell_t\big(\Phi-\Psi\big)z\theta g_1\mathrm{d}W(t).
\end{aligned}
$$

Together the above equality with (A.2)–(A.6) gives the desired identity (3.2). ◻

Appendix B. Proof of Theorem 2.1. For $i=1,2$, recalling the definition of θ_i, one can see that

$$
\begin{aligned}
& \ell_{i,t}=\lambda\eta_{i,t},\quad \ell_{i,x}=\lambda\mu\varphi_i\psi_{i,x},\quad \ell_{i,xx}=\lambda\mu^2\varphi_i\psi_{i,x}^2+\lambda\mu\varphi_i\psi_{i,xx},\\
& \ell_{i,xxx}=\lambda\mu^3\varphi_i\psi_{i,x}^3+3\lambda\mu^2\varphi_i\psi_{i,x}\psi_{i,xx}+\lambda\mu\varphi_i\psi_{i,xxx},
\end{aligned}
$$

and

$$
|\eta_{i,t}|\le CTe^{2\mu|\psi_i|_{C([0,1])}}\varphi_i^{\frac{3}{2}},\quad |\varphi_{i,t}|\le CT\varphi_i^{\frac{3}{2}}.
$$

Notice that $y(0,t)=y(1,t)=0$ on $(0,T)$ and

$$
f_1\in L^2_{\mathbb{F}}(0,T;H_0^2(0,1)),\ \ g_1\in L^2_{\mathbb{F}}(0,T;H^4(0,1)\cap H_0^2(0,1)).
$$

Then

$$
\hat{y}(0,t)=\hat{y}(1,t)=\hat{y}_x(0,t)=\hat{y}_x(1,t)=0\ \text{ on } (0,T).
$$

Choose $\theta=\theta_1$ and $\ell=\ell_1$ in (3.2). Then, by integrating (3.2) in Q and taking expectation, noting that $\theta_1(0)=\theta_1(T)=0$, we have

$$\begin{aligned}
&\mathbb{E}\int_Q \theta_1 I\big(\mathrm{d}\hat{y}+y_{xxxx}\mathrm{d}t\big)\mathrm{d}x\\
&=\mathbb{E}\int_Q\Big[V_x\mathrm{d}t+4\big(\ell_{1,x}\hat{z}\mathrm{d}z_{xx}\big)_x\Big]\mathrm{d}x+\mathbb{E}\int_Q\Big\{I^2+Az^2+Bz_x^2\\
&\qquad+6\ell_{1,xx}\hat{z}_x^2+\Big[\Psi-6D_x+\ell_{1,tt}+12\big(\ell_{1,x}^3-\ell_{1,x}\ell_{1,xx}\big)_x\Big]z_{xx}^2\\
&\qquad+2\ell_{1,xx}z_{xxx}^2+P+U+\big(\ell_{1,tt}-2\ell_{1,xxxx}-2D_x-\Psi\big)\hat{z}^2\Big\}\mathrm{d}x\mathrm{d}t\\
&\quad+\mathbb{E}\int_Q\Big[\ell_{1,t}\big(\Phi-\Psi\big)\big(\mathrm{d}z\big)^2+\ell_{1,t}\big(\mathrm{d}z_{xx}\big)^2+\ell_{1,t}\big(\mathrm{d}\hat{z}\big)^2\\
&\qquad-6\ell_{1,t}\big(\ell_{1,x}^2-\ell_{1,xx}\big)\big(\mathrm{d}z_x\big)^2+4\ell_{1,x}\mathrm{d}\hat{z}\mathrm{d}z_{xxx}+4D\mathrm{d}\hat{z}\mathrm{d}z_x-\Psi\mathrm{d}z\mathrm{d}\hat{z}\Big]\mathrm{d}x.
\end{aligned}\tag{B.1}$$

Now we evaluate the right-hand side of equality (B.1) term by term. For the first one, by recalling the definition of V, and noting that $\psi_{1,x}<0$ in $[0,1]$, we have

$$\begin{aligned}
&\mathbb{E}\int_Q\Big[V_x\mathrm{d}t+4\big(\ell_{1,x}\hat{z}\mathrm{d}z_{xx}\big)_x\Big]\mathrm{d}x\\
&=\mathbb{E}\int_Q\Big[-2\ell_{1,x}z_{xxx}^2+2Dz_{xx}^2-12\big(\ell_{1,x}^3-\ell_{1,x}\ell_{1,xx}\big)z_{xx}^2\Big]_x\mathrm{d}x\mathrm{d}t\\
&=\mathbb{E}\int_0^T\Big\{-2\lambda\mu\varphi_1\psi_{1,x}z_{xxx}^2+\Big[6\lambda\mu\varphi_1\psi_{1,x}\ell_{1,xx}\big(\lambda\mu^2\varphi_1\psi_{1,x}^2+\lambda\mu\varphi_1\psi_{1,xx}\big)\\
&\qquad-10\lambda^3\mu^3\varphi_1^3\psi_{1,x}^3+2\lambda\mu\varphi_1\big(\mu^2\psi_{1,x}^3+3\mu\psi_{1,x}\psi_{1,xx}+\psi_{1,xxx}\big)\Big]z_{xx}^2\Big\}\Big|_0^1\mathrm{d}t\\
&\geq -C\mathbb{E}\int_0^T\Big[\lambda\mu\varphi_1 z_{xxx}^2(0,t)+\lambda^3\mu^3\varphi_1^3z_{xx}^2(0,t)\Big]\mathrm{d}t.
\end{aligned}\tag{B.2}$$

For the second one, from the definitions of A and B, one can get that

$$\begin{aligned}
A&=3\big(\Psi\ell_{1,x}^2-\Psi\ell_{1,xx}\big)_{xx}+\big(\ell_{1,t}\Phi-\ell_{1,t}\Psi\big)_t+2\big(\ell_{1,x}\Phi-\ell_{1,x}\Psi\big)_{xxx}\\
&\quad+\Psi\big(\Phi-\Psi\big)+2\Big[D\big(\Phi-\Psi\big)\Big]_x\\
&=\mathcal{O}(\lambda^5\mu^8\varphi_1^5)+\mathcal{O}_\mu(\lambda^5\varphi_1^6)+\mathcal{O}(\lambda^5\mu^6\varphi_1^5)-9\ell_{1,x}^6\ell_{1,xx}+\mathcal{O}(\lambda^6\mu^8\varphi_1^6)\\
&\quad+\mathcal{O}_\mu(\lambda^5\varphi_1^6)+2(\ell_{1,x}^7)_x+\mathcal{O}(\lambda^6\mu^8\varphi_1^6)+\mathcal{O}_\mu(\lambda^5\varphi_1^6)\\
&=-9\ell_{1,x}^6\ell_{1,xx}+14\ell_{1,x}^6\ell_{1,xx}+\mathcal{O}(\lambda^6\mu^8\varphi_1^6)+\mathcal{O}_\mu(\lambda^5\varphi_1^6)\\
&=5\lambda^7\mu^8\varphi_1^7\psi_{1,x}^8+\mathcal{O}(\lambda^7\mu^7\varphi_1^7)+\mathcal{O}(\lambda^6\mu^8\varphi_1^6)+\mathcal{O}_\mu(\lambda^5\varphi_1^6)\\
&\geq C\lambda^7\mu^8\varphi_1^7,
\end{aligned}\tag{B.3}$$

and

$$\begin{aligned}
B&=12\Big[D\big(\ell_x^2-\ell_{xx}\big)\Big]_x-6\Psi\big(\ell_x^2-\ell_{xx}\big)-6\big(\ell_t\ell_x^2-\ell_t\ell_{xx}\big)_t-6\big(\ell_x\Phi-\ell_x\Psi\big)_x\\
&=60\ell_{1,x}^4\ell_{1,xx}+\mathcal{O}(\lambda^4\mu^6\varphi_1^4)+54\ell_{1,x}^4\ell_{1,xx}-6(\ell_{1,x}^5)_x+\mathcal{O}(\lambda^4\mu^6\varphi_1^4)+\mathcal{O}_\mu(\lambda^3\varphi_1^4)\\
&=84\ell_{1,x}^4\ell_{1,xx}+\mathcal{O}(\lambda^4\mu^6\varphi_1^4)+\mathcal{O}_\mu(\lambda^3\varphi_1^4)\\
&=84\lambda^5\mu^6\varphi_1^5\psi_{1,x}^6+\mathcal{O}(\lambda^5\mu^5\varphi_1^5)+\mathcal{O}(\lambda^4\mu^6\varphi_1^4)+\mathcal{O}_\mu(\lambda^3\varphi_1^4)\\
&\geq C\lambda^5\mu^6\varphi_1^5,
\end{aligned}\tag{B.4}$$

where $\mathcal{O}(\mu^k)$ denotes a function of order μ^k for a sufficiently large μ, and $\mathcal{O}_\mu(\lambda^k)$ denotes a function of order λ^k for a fixed μ and sufficiently large λ. For the third one, by the definitions of Ψ and D, we know

$$\begin{aligned}
&\mathbb{E}\int_Q \Big[\Psi - 6D_x + \ell_{1,tt} + 12\big(\ell_{1,x}^3 - \ell_{1,x}\ell_{1,xx}\big)_x\Big] z_{xx}^2 \mathrm{d}x\mathrm{d}t \\
&= \mathbb{E}\int_Q \big(- 9\ell_{1,x}^2\ell_{1,xx} + 18\ell_{1,x}^2\ell_{1,xx} + 6\ell_{1,xx}^2 + 6\ell_{1,x}\ell_{1,xxx} \\
&\qquad\qquad - 6\ell_{1,xxxx} + \ell_{1,tt}\big) z_{xx}^2 \mathrm{d}x\mathrm{d}t \\
&= \mathbb{E}\int_Q \Big[9\lambda^3\mu^4\varphi_1^3\psi_{1,x}^4 + \mathcal{O}(\lambda^2\mu^4\varphi_1^2) + \mathcal{O}_\mu(\lambda\varphi_1^2) + \mathcal{O}(\lambda^3\mu^3\varphi_1^3)\Big] z_{xx}^2 \mathrm{d}x\mathrm{d}t \\
&\geq C\mathbb{E}\int_Q \lambda^3\mu^4\varphi_1^3 z_{xx}^2 \mathrm{d}x\mathrm{d}t
\end{aligned} \tag{B.5}$$

and

$$\begin{aligned}
&\mathbb{E}\int_Q \big(\ell_{1,tt} - 2\ell_{1,xxxx} - 2D_x - \Psi\big)\hat{z}^2 \mathrm{d}x\mathrm{d}t \\
&= \mathbb{E}\int_Q \big(3\ell_{1,x}^2\ell_{1,xx} + 6\ell_{1,xx}^2 + 6\ell_{1,x}\ell_{1,xxx} + \ell_{1,tt} - 4\ell_{1,xxxx}\big)\hat{z}^2 \mathrm{d}x\mathrm{d}t \\
&= \mathbb{E}\int_Q \Big[3\lambda^3\mu^4\varphi_1^3\psi_{1,x}^4 + \mathcal{O}(\lambda^2\mu^4\varphi_1^2) + \mathcal{O}_\mu(\lambda\varphi_1^2) + \mathcal{O}(\lambda^3\mu^3\varphi_1^3)\Big]\hat{z}^2 \mathrm{d}x\mathrm{d}t \\
&\geq C\mathbb{E}\int_Q \lambda^3\mu^4\varphi_1^3\hat{z}^2 \mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{B.6}$$

For the fourth one, by the Itô formula, one can obtain that

$$\begin{aligned}
&\mathbb{E}\int_Q \Big[\ell_{1,t}\big(\Phi - \Psi\big)\big(\mathrm{d}z\big)^2 - 6\ell_{1,t}\big(\ell_{1,x}^2 - \ell_{1,xx}\big)\big(\mathrm{d}z_x\big)^2 + \ell_{1,t}\big(\mathrm{d}z_{xx}\big)^2 \\
&\qquad + \ell_{1,t}\big(\mathrm{d}\hat{z}\big)^2 + 4\ell_{1,x}\mathrm{d}\hat{z}\mathrm{d}z_{xxx} + 4D\mathrm{d}\hat{z}\mathrm{d}z_x - \Psi\mathrm{d}z\mathrm{d}\hat{z}\Big]\mathrm{d}x \\
&= \mathbb{E}\int_Q \Big\{\ell_{1,t}\big(\Phi - \Psi\big)\big(\theta_1 g_1\big)^2 - 6\ell_{1,t}\big(\ell_{1,x}^2 - \ell_{1,xx}\big)\Big[\big(\theta_1 g_1\big)_x\Big]^2 \\
&\qquad + \ell_{1,t}\Big[\big(\theta_1 g_1\big)_{xx}\Big]^2 + \theta_1^2\ell_{1,t}\big(\ell_{1,t}g_1 + g_2\big)^2 + 4\theta_1 D\big(\ell_{1,t}g_1 + g_2\big)\big(\theta_1 g_1\big)_x \\
&\qquad + 4\theta_1\ell_{1,x}\big(\ell_{1,t}g_1 + g_2\big)\big(\theta_1 g_1\big)_{xxx} - \Psi\theta_1^2 g_1\big(\ell_{1,t}g_1 + g_2\big)\Big\}\mathrm{d}x\mathrm{d}t \\
&\geq -C\mathbb{E}\int_Q \theta_1^2\Big[|\ell_{1,t}|\big(\ell_{1,x}^4 g_1^2 + \ell_{1,x}^2 g_{1,x}^2 + g_{1,xx}^2 + \ell_{1,t}^2 g_1^2 + g_2^2 \\
&\qquad + |\ell_{1,x}^3 g_1 g_{1,x}| + |\ell_{1,x}^3 g_1 g_{1,x}| + \ell_{1,x}^2|g_1 g_{1,xx}| + |\ell_{1,x} g_1 g_{1,xxx}|\big) \\
&\qquad + |\ell_{1,x}^3 g_{1,x} g_2| + \ell_{1,x}^4|g_1 g_2|\Big]\mathrm{d}x\mathrm{d}t \\
&\geq -C\mathbb{E}\int_Q \theta_1^2\big(\lambda^6\varphi_1^6 g_1^2 + \lambda^4\varphi_1^4 g_{1,x}^2 + \lambda^2\varphi_1^2 g_{1,xx}^2 + g_{1,xxx}^2 + \lambda^2\mu^8\varphi_1^2 g_2^2\big)\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{B.7}$$

For the fifth one, from the definitions of U and P, there exist $\lambda_1, \mu_1 \geq 1$ such that for any $\lambda \geq \lambda_1$ and $\mu \geq \mu_1$ one can get

$$\begin{aligned}
&\left|\mathbb{E}\int_Q U\mathrm{d}x\mathrm{d}t\right| \\
&\leq C\mathbb{E}\int_Q \theta_1\Big[|\ell_{1,t}|\big(|\ell_{1,t}\hat{z}f_1| + |\ell_{1,x}z_{xxx}f_1| + |\ell_{1,x}^3 z_x f_1| + \ell_{1,x}^4|zf_1|\big) \\
&\qquad + \ell_{1,x}^4|\hat{z}f_1| + |\ell_{1,x}^3\hat{z}f_{1,x}| + \ell_{1,x}^2|\ell_{1,t}z_x f_{1,x}| + |\ell_{1,t}|\big(\ell_{1,x}^2|z_{xx}f_1| \\
&\qquad + |\ell_{1,x}z_{xx}f_{1,x}| + |z_{xx}f_{1,xx}|\big) + |\ell_{1,xx}\hat{z}f_{1,xx}| \\
&\qquad + |\ell_{1,x}|\big(\ell_{1,x}^2|\hat{z}_x f_1| + |\ell_{1,x}\hat{z}_x f_{1,x}| + |\hat{z}_x f_{1,xx}|\big)\Big]\mathrm{d}x\mathrm{d}t \\
&\leq C\mathbb{E}\int_Q \theta_1^2\big(\lambda^6\mu^6\varphi_1^6 f_1^2 + \lambda^4\mu^4\varphi_1^4 f_{1,x}^2 + \lambda^2\mu^2\varphi_1^2 f_{1,xx}^2\big)\mathrm{d}x\mathrm{d}t \\
&\quad + C\mathbb{E}\int_Q \big(\lambda^4\mu^2\varphi_1^5 z^2 + \lambda^2\mu^2\varphi_1^3 z_x^2 + \varphi_1 z_{xx}^2 + z_{xxx}^2 + \lambda^2\mu^2\varphi_1^2\hat{z}^2 + \hat{z}_x^2\big)\mathrm{d}x\mathrm{d}t
\end{aligned} \tag{B.8}$$

and

$$\begin{aligned}
&\left|\mathbb{E}\int_Q P\mathrm{d}x\mathrm{d}t\right| \\
&\leq C\mathbb{E}\int_Q \Big(\lambda\mu\varphi_1^{\frac{3}{2}}|\hat{z}_x z_{xx}| + \lambda\mu\varphi_1^{\frac{3}{2}}|\hat{z}z_{xxx}| + \lambda^3\mu^5\varphi_1^3|zz_{xxx}| \\
&\qquad + \lambda^3\mu^5\varphi_1^3|z_x z_{xx}| + \lambda^3\mu^4\varphi_1^{\frac{7}{2}}|\hat{z}z| + \lambda^3\mu^3\varphi_1^{\frac{7}{2}}|\hat{z}z_x|\Big)\mathrm{d}x\mathrm{d}t \\
&\leq C\mathbb{E}\int_Q \Big[\lambda^6\mu^8\varphi_1^6 z^2 + \lambda^4\mu^6\varphi_1^5 z_x^2 + \lambda^2\mu^4\varphi_1^3\big(z_{xx}^2 + \hat{z}^2\big) + \mu^2 z_{xxx}^2 + \hat{z}_x^2\Big]\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{B.9}$$

By (B.1)–(B.9), there exist two sufficiently large $\lambda_2, \mu_2 \geq 1$, such that for any $\lambda \geq \lambda_2$ and $\mu \geq \mu_2$ it holds that

$$\begin{aligned}
&\mathbb{E}\int_Q \Big[\lambda^7\mu^8\varphi_1^7 z^2 + \lambda^5\mu^6\varphi_1^5 z_x^2 + \lambda^3\mu^4\varphi_1^3\big(z_{xx}^2 + \hat{z}^2\big) + \lambda\mu^2\varphi_1\big(z_{xxx}^2 + \hat{z}_x^2\big)\Big]\mathrm{d}x\mathrm{d}t \\
&\leq C\mathbb{E}\int_Q \theta_1^2\big(\lambda^6\mu^6\varphi_1^6 f_1^2 + \lambda^4\mu^4\varphi_1^4 f_{1,x}^2 + \lambda^2\mu^2\varphi_1^2 f_{1,xx}^2 + f_2^2 \\
&\qquad + \lambda^6\varphi_1^6 g_1^2 + \lambda^4\varphi_1^4 g_{1,x}^2 + \lambda^2\varphi_1^2 g_{1,xx}^2 + g_{1,xxx}^2 + \lambda^2\mu^8\varphi_1^2 g_2^2\big)\mathrm{d}x\mathrm{d}t \\
&\quad + C\mathbb{E}\int_0^T \Big[\lambda\mu\varphi_1 z_{xxx}^2(0,t) + \lambda^3\mu^3\varphi_1^3 z_{xx}^2(0,t)\Big]\mathrm{d}t.
\end{aligned}$$

From

$$y_x = \theta_1^{-1}\big(z_x - \ell_{1,x}z\big) = \theta_1^{-1}\big(z_x - \lambda\mu\varphi_1\psi_{1,x}z\big)$$

and

$$z_x = \theta_1\big(y_x + \ell_{1,x}y\big) = \theta_1\big(y_x + \lambda\mu\varphi_1\psi_{1,x}y\big),$$

we get that

$$\frac{1}{C}\theta_1^2\big(y_x^2 + \lambda^2\mu^2\varphi_1^2 y^2\big) \leq z_x^2 + \lambda^2\mu^2\varphi_1^2 z^2 \leq C\theta_1^2\big(y_x^2 + \lambda^2\mu^2\varphi_1^2 y^2\big).$$

Likewise, z_{xx} and y_{xx}, z_{xxx} and y_{xxx}, $\hat{z}$ and $\hat{y}$, and $\hat{z}_x$ and $\hat{y}_x$ have similar estimates. Therefore, there exist $\mu_3 > 0$ and $\lambda_3 = \lambda_3(\mu)$, such that for any $\mu \geq \mu_3$ and $\lambda \geq \lambda_3(\mu)$

it follows that

$$\begin{aligned}
&\mathbb{E}\int_Q \theta_1^2\Big[\lambda^7\mu^8\varphi_1^7 y^2+\lambda^5\mu^6\varphi_1^5 y_x^2+\lambda^3\mu^4\varphi_1^3\big(y_{xx}^2+\hat{y}^2\big)+\lambda\mu^2\varphi_1\big(y_{xxx}^2+\hat{y}_x^2\big)\Big]\mathrm{d}x\mathrm{d}t\\
&\le C\mathbb{E}\int_Q \theta_1^2\big(\lambda^6\mu^6\varphi_1^6 f_1^2+\lambda^4\mu^4\varphi_1^4 f_{1,x}^2+\lambda^2\mu^2\varphi_1^2 f_{1,xx}^2+f_2^2\\
&\qquad+\lambda^6\varphi_1^6 g_1^2+\lambda^4\varphi_1^4 g_{1,x}^2+\lambda^2\varphi_1^2 g_{1,xx}^2+g_{1,xxx}^2+\lambda^2\mu^8\varphi_1^2 g_2^2\big)\mathrm{d}x\mathrm{d}t\\
&\quad+C\mathbb{E}\int_0^T\Big[\lambda\mu\varphi_1 z_{xxx}^2(0,t)+\lambda^3\mu^3\varphi_1^3 z_{xx}^2(0,t)\Big]\mathrm{d}t.
\end{aligned}$$

Thus, the desired result (2.2) in Theorem 2.1 is obtained. ◻

Appendix C. Proof of Theorem 2.2. Take $\theta=\theta_2$ and $\ell=\ell_2$ in Lemma 3.1. Note that $\psi_{2,x}(0)>0$ and $\psi_{2,x}(1)<0$; then, by integrating (3.2) on Q and taking expectation, for sufficiently large λ and μ, we have

$$\begin{aligned}
&\mathbb{E}\int_Q\Big[V_x\mathrm{d}t+4\big(\ell_{2,x}\hat{z}\mathrm{d}z_{xx}\big)_x\Big]\mathrm{d}x\\
&=\mathbb{E}\int_0^T\Big\{-2\lambda\mu\varphi_2\psi_{2,x}z_{xxx}^2+\Big[6\lambda\mu\varphi_2\psi_{2,x}\ell_{2,xx}\big(\lambda\mu^2\varphi_2\psi_{2,x}^2+\lambda\mu\varphi_2\psi_{2,xx}\big)\\
&\qquad-10\lambda^3\mu^3\varphi_2^3\psi_{2,x}^3+2\lambda\mu\varphi_2\big(\mu^2\psi_{2,x}^3+3\mu\psi_{2,x}\psi_{2,xx}+\psi_{2,xxx}\big)\Big]z_{xx}^2\Big\}\Big|_0^1\mathrm{d}t\\
&=\mathbb{E}\int_0^T\Big\{-2\lambda\mu\varphi_2\psi_{2,x}z_{xxx}^2+\Big[-2\lambda^3\mu^3\varphi_2^3\psi_{2,x}^3+\mathcal{O}\big(\lambda^2\mu^2\varphi_2^2\big)\Big]z_{xx}^2\Big\}\Big|_0^1\mathrm{d}t\\
&\ge 0.
\end{aligned}\tag{C.1}$$

Similar to the derivation of (B.3)–(B.9) and combining with (C.1), there are two positive constants $\lambda_4,\mu_4\ge 1$ such that for any $\lambda\ge\lambda_4$ and $\mu\ge\mu_4$ it holds that

$$\begin{aligned}
&\mathbb{E}\int_Q\Big[\lambda^7\mu^8\varphi_2^7 z^2+\lambda^5\mu^6\varphi_2^5 z_x^2+\lambda^3\mu^4\varphi_2^3\big(z_{xx}^2+\hat{z}^2\big)+\lambda\mu^2\varphi_2\big(z_{xxx}^2+\hat{z}_x^2\big)\Big]\mathrm{d}x\mathrm{d}t\\
&\le C\mathbb{E}\int_Q\theta_2^2\big(\lambda^6\mu^6\varphi_2^6 f_1^2+\lambda^4\mu^4\varphi_2^4 f_{1,x}^2+\lambda^2\mu^2\varphi_2^2 f_{1,xx}^2+\lambda^6\varphi_2^6 g_1^2+\lambda^4\varphi_2^4 g_{1,x}^2\\
&\qquad+\lambda^2\varphi_2^2 g_{1,xx}^2+g_{1,xxx}^2+\lambda^2\mu^8\varphi_2^2 g_2^2+f_2^2\big)\mathrm{d}x\mathrm{d}t+C\mathbb{E}\int_0^T\int_{G_1}\Big[\lambda^7\mu^8\varphi_2^7 z^2\\
&\qquad+\lambda^5\mu^6\varphi_2^5 z_x^2+\lambda^3\mu^4\varphi_2^3\big(z_{xx}^2+\hat{z}^2\big)+\lambda\mu^2\varphi_2\big(z_{xxx}^2+\hat{z}_x^2\big)\Big]\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

By $z=\theta_2 y$, we have

$$\frac{1}{C}\theta_2^2\big(y_x^2+\lambda^2\mu^2\varphi_2^2 y^2\big)\le z_x^2+\lambda^2\mu^2\varphi_2^2 z^2\le C\theta_2^2\big(y_x^2+\lambda^2\mu^2\varphi_2^2 y^2\big).$$

Likewise, z_{xx} and y_{xx}, z_{xxx} and y_{xxx}, $\hat{z}$ and $\hat{y}$, and $\hat{z}_x$ and $\hat{y}_x$ have similar estimates. Therefore, there exist $\mu_5>0$ and $\lambda_5=\lambda_5(\mu)$, such that for any $\mu\ge\mu_5$ and $\lambda\ge\lambda_5$

it follows that

(C.2)
$$\begin{aligned}
&\mathbb{E}\int_Q \theta_2^2\Big[\lambda^7\mu^8\varphi_2^7 y^2+\lambda^5\mu^6\varphi_2^5 y_x^2+\lambda^3\mu^4\varphi_2^3\big(y_{xx}^2+\hat{y}^2\big)+\lambda\mu^2\varphi_2\big(y_{xxx}^2+\hat{y}_x^2\big)\Big]\mathrm{d}x\mathrm{d}t\\
&\le C\mathbb{E}\int_Q \theta_2^2\big(\lambda^6\mu^6\varphi_2^6 f_1^2+\lambda^4\mu^4\varphi_2^4 f_{1,x}^2+\lambda^2\mu^2\varphi_2^2 f_{1,xx}^2+\lambda^6\varphi_2^6 g_1^2+\lambda^4\varphi_2^4 g_{1,x}^2\\
&\quad+\lambda^2\varphi_2^2 g_{1,xx}^2+g_{1,xxx}^2+f_2^2+\lambda^2\mu^8\varphi_2^2 g_2^2\big)\mathrm{d}x\mathrm{d}t+C\mathbb{E}\int_0^T\int_{G_1}\theta_2^2\Big[\lambda^7\mu^8\varphi_2^7 y^2\\
&\quad+\lambda^5\mu^6\varphi_2^5 y_x^2+\lambda^3\mu^4\varphi_2^3\big(y_{xx}^2+\hat{y}^2\big)+\lambda\mu^2\varphi_2\big(y_{xxx}^2+\hat{y}_x^2\big)\Big]\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Next, the estimates on $\mathbb{E}\int_0^T\int_{G_1}\theta_2^2\lambda^3\mu^4\varphi_2^3\hat{y}^2\mathrm{d}x\mathrm{d}t$ and $\mathbb{E}\int_0^T\int_{G_1}\theta_2^2\lambda\mu^2\varphi_2\hat{y}_x^2\mathrm{d}x\mathrm{d}t$ are given. To this aim, choose a function $\tilde{\zeta}\in C_0^\infty(G_0)$ satisfying $0\le\tilde{\zeta}\le 1$ in G_0, $\tilde{\zeta}\equiv 1$ in G_1. Note that

$$\mathrm{d}\big(\theta_2^2\lambda^3\mu^4\varphi_2^3\tilde{\zeta}y\hat{y}\big)=\theta_2^2\lambda^3\mu^4\varphi_2^3\tilde{\zeta}\big(y\mathrm{d}\hat{y}+\hat{y}\mathrm{d}y+\mathrm{d}y\mathrm{d}\hat{y}\big)+\big(\theta_2^2\lambda^3\mu^4\varphi_2^3\tilde{\zeta}\big)_t y\hat{y}\mathrm{d}t.$$

Then, by a simple calculation and Hölder's inequality, we have that for any $\rho>0$,

(C.3)
$$\begin{aligned}
&\mathbb{E}\int_Q \theta_2^2\lambda^3\mu^4\varphi_2^3\tilde{\zeta}\hat{y}^2\mathrm{d}x\mathrm{d}t\\
&=\mathbb{E}\int_Q\Big[\theta_2^2\lambda^3\mu^4\varphi_2^3\tilde{\zeta}y_{xx}^2+\big(\theta_2^2\lambda^3\mu^4\varphi_2^3\tilde{\zeta}\big)_x y_x y_{xx}-\big(\theta_2^2\lambda^3\mu^4\varphi_2^3\tilde{\zeta}\big)_x y y_{xxx}\\
&\qquad-\theta_2^2\lambda^3\mu^4\varphi_2^3\tilde{\zeta}\big(yf_2+\hat{y}f_1+g_1g_2\big)-\big(\theta_2^2\lambda^3\mu^4\varphi_2^3\tilde{\zeta}\big)_t y\hat{y}\Big]\mathrm{d}x\mathrm{d}t\\
&\le C\mathbb{E}\int_Q\theta_2^2\tilde{\zeta}\big(\lambda^6\mu^8\varphi_2^6 y^2+\lambda^5\mu^6\varphi_2^3 y_x^2+\lambda^3\mu^4\varphi_2^3 y_{xx}^2+\lambda\mu^2\varphi_2 y_{xxx}^2+f_2^2\\
&\qquad+\lambda^3\mu^4\varphi_2^3 f_1^2+\lambda^6\varphi_2^6 g_1^2+\lambda^2\mu^8\varphi_2^2 g_2^2\big)\mathrm{d}x\mathrm{d}t+\rho\mathbb{E}\int_Q\theta_2^2\lambda^3\mu^4\varphi_2^3\tilde{\zeta}\hat{y}^2\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Similarly, by $\mathrm{d}\big(\theta_2^2\lambda\mu^2\varphi_2\tilde{\zeta}y_{xx}\hat{y}\big)$ and Itô's formula, it can be seen that for any $\rho>0$,

(C.4)
$$\begin{aligned}
&\mathbb{E}\int_Q\theta_2^2\lambda\mu^2\varphi_2\tilde{\zeta}\hat{y}_x^2\mathrm{d}x\mathrm{d}t=\mathbb{E}\int_Q\Big[\theta_2^2\lambda\mu^2\varphi_2\tilde{\zeta}y_{xxx}^2-\big(\theta_2^2\lambda\mu^2\varphi_2\tilde{\zeta}\big)_x y_{xx}y_{xxx}\\
&\qquad-\theta_2^2\lambda\mu^2\varphi_2\tilde{\zeta}\big(y_{xx}f_2+f_{1,xx}\hat{y}+g_{1,xx}g_2\big)-\big(\theta_2^2\lambda\mu^2\varphi_2\tilde{\zeta}\big)_t y_{xx}\hat{y}\Big]\mathrm{d}x\mathrm{d}t\\
&\le C\mathbb{E}\int_Q\theta_2^2\tilde{\zeta}\Big[\lambda^3\mu^4\varphi_2^3 y_{xx}^2+\lambda\mu^2\varphi_2 y_{xxx}^2+f_2^2+\lambda\mu^2\varphi_2\big(f_{1,xx}^2+\hat{y}^2\big)\\
&\qquad+\lambda^2\varphi_2^2 g_{1,xx}^2+\lambda^2\mu^8\varphi_2^2 g_2^2\Big]\mathrm{d}x\mathrm{d}t+\rho\mathbb{E}\int_Q\theta_2^2\lambda\mu^2\varphi_2\tilde{\zeta}\hat{y}_x^2\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Finally, combining (C.2)–(C.4), one can get the desired estimate (2.3). ☐

Acknowledgments. The authors would like to thank the editor and the anonymous referees for their valuable comments and suggestions for improving the paper.

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